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Functions of unitaries with $\mathcal{S}^p$-perturbations for non-continuously differentiable functions

by

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Abstract. Consider a function $f : \mathbb{T} \to \mathbb{C}$, n-times differentiable on $\mathbb{T}$ and such that its nth derivative $f^{(n)}$ is bounded but not necessarily continuous. Let $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ be a function taking values in the set of unitary operators on some separable Hilbert space $\mathcal{H}$. Let $1 < p < \infty$ and let $\mathcal{S}^p(\mathcal{H})$ be the Schatten class of order p on $\mathcal{H}$. If $\tilde{U} : \mathbb{R} \ni t \mapsto U(t) - U(0)$ is n-times $\mathcal{S}^p$-differentiable on $\mathbb{R}$, we show that the operator-valued function $\varphi : \mathbb{R} \ni t \mapsto f(U(t)) - f(U(0)) \in \mathcal{S}^p(\mathcal{H})$ is n-times differentiable on $\mathbb{R}$ as well. This theorem is optimal and extends several results related to the differentiability of functions of unitaries. The derivatives of φ are given in terms of multiple operator integrals, and a formula and $\mathcal{S}^p$-estimates for the Taylor remainders of φ are provided.

1. Introduction. Let $\mathcal{H}$ be a separable complex Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denote the Banach space of bounded operators on $\mathcal{H}$, and let $\mathcal{U}(\mathcal{H})$ be the subset of unitary operators. For any $1 < p < \infty$, $\mathcal{S}^p(\mathcal{H})$ will denote the Schatten class of order p on $\mathcal{H}$, that is, the Banach space defined by

$$\mathcal{S}^p(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid \|A\|_p := \mathrm{Tr}(|A|^p)^{1/p} < \infty\}.$$

The study of differentiability of operator functions was initiated in [11]. Since then, it has attracted a lot of attention and significant refinements have been obtained in [1, 3, 4, 6, 7, 12, 15–17, 19, 20, 24, 28]. This study has often been motivated by problems in perturbation theory. For instance, various fruitful efforts to prove the existence of spectral shift functions [18, 21, 22, 26] naturally led to the question of the existence and the representation of the derivatives of

$$\varphi : \mathbb{R} \ni t \mapsto f(e^{itA}U) - f(U),$$

where $A = A^* \in \mathcal{B}(\mathcal{H})$, $U \in \mathcal{U}(\mathcal{H})$ and $f : \mathbb{T} \to \mathbb{C}$ is a function defined on $\mathbb{T}$, the unit circle of $\mathbb{C}$. In [24], the authors proved that if f belongs to the Besov class $B^n_{\infty 1}(\mathbb{T})$, $n \geq 2$, the nth order derivative of φ exists in the operator

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norm. For the Schatten classes, it was proved in [5] that if $1 < p < \infty$ and $A \in \mathcal{S}^p(\mathcal{H})$, then, under the assumption $f \in C^n(\mathbb{T})$, the function φ is n-times continuously $\mathcal{S}^p$-differentiable on $\mathbb{R}$. In fact, stronger results hold [5, Theorem 3.3]. The common denominator in all these results is the use of the theory of multiple operator integrals, which can be seen as the measurable counterparts of Schur multipliers. In particular, the derivatives of φ can be expressed as multiple operator integrals or a linear combination of them, with respect to the divided differences of f. See for instance [25, Theorem 3.7] for the finite-dimensional case and [27, Theorem 5.3.4] or [5, Theorem 3.5] for the infinite-dimensional case.

In the selfadjoint case, more is known. The analogous question is to investigate under which assumptions on $g : \mathbb{R} \to \mathbb{C}$, the function

$$\psi : \mathbb{R} \ni t \mapsto g(A + tK) - g(A)$$

is differentiable, where A and K are selfadjoint with K bounded. When $g \in C^n(\mathbb{R})$ with bounded derivatives and $K \in \mathcal{S}^p$ with $1 < p < \infty$, it is known that ψ is $\mathcal{S}^p$-differentiable with continuous derivatives [7, 17]. In fact, the existence of ψ' in the $\mathcal{S}^p$-norm holds when the assumptions on g are relaxed. Indeed, one of the striking results is given in [15], where the authors proved that the condition "g differentiable on $\mathbb{R}$ with bounded derivative" ensures the differentiability of ψ in the $\mathcal{S}^p$-norm. This is a fundamental difference from the $\mathcal{B}(\mathcal{H})$ case, since it is known that the stronger condition "$g \in C^1(\mathbb{R})$ with bounded derivative" is not sufficient for the existence of ψ' in the operator norm [13]. A generalization of the aforementioned result for the higher order differentiability of ψ has been established in [6], where it was shown that if g is n-times differentiable with bounded derivatives $g', \ldots, g^{(n)}$, then so is ψ. It appears that the corresponding result for functions of unitaries was not known, even in the case $n = 1$ and in the Hilbert–Schmidt case $\mathcal{S}^2(\mathcal{H})$. Namely, if we drop the assumption of continuity of the derivative of $f : \mathbb{T} \to \mathbb{C}$, do we have the differentiability of φ in $\mathcal{S}^2(\mathcal{H})$ or even in $\mathcal{S}^p(\mathcal{H})$?

In this paper, we solve this last question in two ways: first by requiring the minimal assumptions on f, and secondly by obtaining the nth order differentiability for the associated operator function. We prove (see Theorem 5.1) that if $1 < p < \infty$, f is an n-times differentiable function on $\mathbb{T}$ with a bounded nth derivative $f^{(n)}$ and $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ is such that $\tilde{U} : \mathbb{R} \ni t \mapsto U(t) - U(0) \in \mathcal{S}^p(\mathcal{H})$ is n-times differentiable, then the operator-valued function

$$\varphi : \mathbb{R} \ni t \mapsto f(U(t)) - f(U(0)) \in \mathcal{S}^p(\mathcal{H})$$

is n-times differentiable on $\mathbb{R}$. Moreover, if U has bounded derivatives, then so does φ. We show that the explicit formulas for the derivatives of φ given as a sum of multiple operator integrals, obtained with stronger assumptions in [5, 25, 27], also hold true at the degree of generality aimed at in this paper. Note that this result is optimal: it is clear that if φ is differentiable for every differentiable function U, then f itself must be differentiable. In particular, this paper settles the question of $\mathcal{S}^p$-differentiability for functions of unitaries. Additionally, we explain in Remark 5.3 how to obtain a representation of the Taylor remainder

$$R_{n,f,U}(t) := f(U(t)) - f(U(0)) - \sum_{m=1}^{n-1} \frac{1}{m!} \varphi^{(m)}(0),$$

as well as an estimate of its $\mathcal{S}^p$-norm in the case $U(t) = e^{itA}U$.

To achieve our results, we first have to establish important properties of multiple operator integrals, such as their boundedness on Schatten classes when they are associated to divided differences, and some of their properties that will be suitable to study the differentiability of operator functions. Some of the properties are similar to those in [5]; however, in this more general setting, the proofs will require more care. In particular, our approach uses the construction of multiple operator integrals as defined in [8], which is appropriate for our study as it is very general. Next, we will show that with the help of a Cayley transform, we can use the selfadjoint analogue of our main result, proved in [6], to obtain our result in a particular case. This step is crucial and this is where the biggest differences appear between the case when f only has a bounded nth derivative, and the case when f has more regularity such as $f \in C^n(\mathbb{T})$. In the latter case, one can approximate f and its derivatives uniformly (which yields stronger results), while when the assumptions are relaxed, the approach of [6, 15] rests on the approximation of the operators appearing in the $\mathcal{S}^p$-perturbation. Finally, the main result, Theorem 5.1, will follow from a careful approximation of the path of unitaries.

The paper is organized as follows: In Section 2, we give the definition of the divided differences of a function f and show that they can be approximated by more regular functions in Lemmas 2.1 and 2.2. In Section 3, we recall the definition of multiple operator integrals and establish some of their properties such as $\mathcal{S}^p$-boundedness in Theorem 3.3 and an important perturbation formula in Proposition 3.5. In Section 4, we generalize the main result of [6] to be able to apply it in Proposition 4.4, which is a weaker version of our main result. Finally, Section 5 is dedicated to the proof of Theorem 5.1. The proof will require two auxiliary results, Proposition 5.4 and Lemma 5.5, which are the first steps towards an approximation argument used in the proof of our main result.

Notations and conventions

• Whenever Z is a set and $W \subset Z$ a subset, we let $\chi_W : Z \to \{0,1\}$ denote the characteristic function of W.

• As recalled at the beginning of the introduction, $\mathcal{S}^p(\mathcal{H})$ will denote the p-Schatten class on a complex separable Hilbert space $\mathcal{H}$, and $\mathcal{S}^p_{\rm sa}(\mathcal{H})$ will be the subset of selfadjoint operators in $\mathcal{S}^p(\mathcal{H})$. The $\mathcal{S}^p$-norm of an element $K \in \mathcal{S}^p(\mathcal{H})$ is denoted by $\|K\|_p$.

• Similarly, $\mathcal{B}(\mathcal{H})$ is the Banach space of bounded linear operators $A : \mathcal{H} \to \mathcal{H}$ equipped with the operator norm, denoted by $\|A\|$. We let $\mathcal{B}_{\rm sa}(\mathcal{H})$ denote the subset of bounded selfadjoint operators on $\mathcal{H}$.

• Let $f : \mathbb{T} \to \mathbb{C}$ be a function. The derivative of f at $z_0 \in \mathbb{T}$ is the limit

$$f'(z_0) := \lim_{z\in\mathbb{T},\, z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1.1}$$

provided it exists.

• If $\varphi : \mathbb{R} \to \mathcal{S}^p(\mathcal{H})$ is an $\mathcal{S}^p(\mathcal{H})$-valued function, we will say that φ is differentiable at $s \in \mathbb{R}$ if the limit

$$\varphi'(s) := \lim_{t\to s} \frac{\varphi(t) - \varphi(s)}{t - s}$$

exists in $\mathcal{S}^p(\mathcal{H})$. In that case, $\varphi'(s) \in \mathcal{S}^p(\mathcal{H})$.

• If $T \in \mathcal{B}(\mathcal{H})$, we let $\sigma(T)$ denote the spectrum of T. In particular, if $T \in \mathcal{U}(\mathcal{H})$, then $\sigma(T) \subset \mathbb{T}$.

• For any $k \in \mathbb{N}$, we will use the notation $(T)^k = \underbrace{T, \dots, T}_{k}$.

• Let $n \in \mathbb{N}$ and let $X_1, \dots, X_n, Y$ be Banach spaces. We denote by $\mathcal{B}_n(X_1 \times \cdots \times X_n, Y)$ the space of bounded n-linear operators $T : X_1 \times \cdots \times X_n \to Y$, equipped with the norm

$$\|T\|_{\mathcal{B}_n(X_1\times\cdots\times X_n, Y)} := \sup_{\|x_i\|\leq 1,\, 1\leq i\leq n} \|T(x_1, \dots, x_n)\|.$$

We will sometimes write $\|T\|$ for the norm of T when no confusion can occur. In the case when $X_1 = \cdots = X_n = Y$, we will simply denote this space by $\mathcal{B}_n(Y)$. Finally, note that $\mathcal{B}_n(\mathcal{S}^2(\mathcal{H}))$ is a dual space; see [8, Section 3.1] for details.

2. Divided differences and approximation.

2. Divided differences and approximation. In this section, we first recall the definition of the divided differences of a function f and their properties. Then, we will give the construction of two sequences of elements of $C^n(\mathbb{T})$ which approximate, in a certain sense, the divided differences of a function f with bounded nth derivative. Both constructions have advantages and disadvantages, as explained before each statement.

Let $f : \mathbb{T} \to \mathbb{C}$ be a function defined on $\mathbb{T}$. We define its divided difference $f^{[n]} : \mathbb{T}^{n+1} \to \mathbb{C}$ recursively as follows. First, by convention $f^{[0]} = f$. Next, if f is differentiable, $f^{[1]} : \mathbb{T}^2 \to \mathbb{C}$ is defined by

$$f^{[1]}(\lambda_1, \lambda_2) := \begin{cases} \frac{f(\lambda_1)-f(\lambda_2)}{\lambda_1-\lambda_2} & \text{if } \lambda_1 \neq \lambda_2, \\ f'(\lambda_1) & \text{if } \lambda_1 = \lambda_2, \end{cases} \qquad \lambda_1, \lambda_2 \in \mathbb{T}.$$

Now, let $n \in \mathbb{N}$ and assume that f is n-times differentiable on $\mathbb{T}$. The nth divided difference $f^{[n]} : \mathbb{T}^{n+1} \to \mathbb{C}$ is defined by

$$f^{[n]}(\lambda_1, \lambda_2, \dots, \lambda_{n+1}) := \begin{cases} \frac{f^{[n-1]}(\lambda_1,\lambda_3,\dots,\lambda_{n+1})-f^{[n-1]}(\lambda_2,\lambda_3,\dots,\lambda_{n+1})}{\lambda_1-\lambda_2} & \text{if } \lambda_1 \neq \lambda_2, \\ \partial_1 f^{[n-1]}(\lambda_1, \lambda_3, \dots, \lambda_{n+1}) & \text{if } \lambda_1 = \lambda_2, \end{cases}$$

for all $\lambda_1, \dots, \lambda_{n+1} \in \mathbb{T}$, where ∂_1 stands for the partial derivative with respect to the first variable.

The function $f^{[n]}$ is symmetric in the $n+1$ variables $(\lambda_1, \dots, \lambda_{n+1})$, it is measurable, and $f^{[n]}$ is bounded if and only if $f^{(n)}$ is bounded. Indeed, it follows from [10, Theorem 2.1] that there exists a constant d_n such that

$$\|f^{[n]}\|_{L^\infty(\mathbb{T}^{n+1})} \leq d_n \|f^{(n)}\|_{L^\infty(\mathbb{T})}. \tag{2.1}$$

In [10], the estimate for $|f^{[n]}(\lambda_1, \dots, \lambda_{n+1})|$ was obtained for distinct λ_i, but when f is n-times differentiable, the same inequality readily extends to every point of $\mathbb{T}^{n+1}$.

In the following, we give the first construction of a sequence $(f_j)_j$ which will approximate the derivatives of f and its divided differences. This construction will allow us to obtain a satisfactory bound for the nth divided difference of f_j, which in turn will allow us to get a certain bound in Theorem 3.3.

LEMMA 2.1. *Let $n \in \mathbb{N}$ and let $f : \mathbb{T} \to \mathbb{C}$ be an n-times differentiable function such that $f^{(n)}$ is bounded. Then there exists a sequence $(f_j)_j$ of trigonometric polynomials on $\mathbb{T}$ such that:*

(1) *For every $1 \leq k \leq n-1$, the sequence $(f_j^{[k]})_j$, is uniformly convergent to $f^{[k]}$ on $\mathbb{T}^{k+1}$.*
(2) *The sequence $(f_j^{[n]})_j$ is pointwise convergent to $f^{[n]}$ on the set*
$$\mathbb{T}^{n+1} \setminus \{(\lambda_1, \dots, \lambda_{n+1}) \mid \lambda_1 = \dots = \lambda_{n+1}\}.$$
(3) *For every j,*
$$\|f_j^{[n]}\|_{L^\infty(\mathbb{T}^{n+1})} \leq d_n \|f_j^{(n)}\|_{L^\infty(\mathbb{T})} \leq d_n \|f^{(n)}\|_{L^\infty(\mathbb{T})},$$
where d_n is the constant of (2.1).

Proof. For every $j \in \mathbb{N}$, define $f_j := f * F_j$ where F_j is the Fejér kernel, that is,

$$\forall z = e^{i\theta} \in \mathbb{T}, \quad f_j(z) = \int_0^{2\pi} f(e^{it}) F_j(\theta - t) \, \frac{dt}{2\pi} = \int_0^{2\pi} f(e^{i(\theta-t)}) F_j(t) \, \frac{dt}{2\pi}.$$

For every j, f_j is a trigonometric polynomial. Moreover, since $f^{(n)}$ is bounded, it is well-known that f_j is n-times differentiable on $\mathbb{T}$ and for every $1 \leq k \leq n$,

$$\forall z = e^{i\theta} \in \mathbb{T}, \quad f_j^{(k)}(z) = \int_0^{2\pi} f^{(k)}(e^{i(\theta - t)})e^{-ikt} F_j(t) \, \frac{dt}{2\pi} = (f^{(k)} * F_{j,k})(z),$$

where $F_{j,k}(t) = e^{-ikt} F_j(t)$. In particular, according to (2.1) and by Young's inequality, we have, for every $1 \leq k \leq n$,

$$\begin{aligned} \|f_j^{[k]}\|_{L^\infty(\mathbb{T}^{k+1})} \leq d_k \|f_j^{(k)}\|_{L^\infty(\mathbb{T})} &\leq d_k \|f^{(k)}\|_{L^\infty(\mathbb{T})} \|F_{j,k}\|_{L^1(\mathbb{T})} \\ &= d_k \|f^{(k)}\|_{L^\infty(\mathbb{T})} \|F_j\|_{L^1(\mathbb{T})} \\ &= d_k \|f^{(k)}\|_{L^\infty(\mathbb{T})}. \end{aligned}$$

Next, it is a classical fact that for every $1 \leq k \leq n-1$,

$$f_j^{(k)} \xrightarrow[j \to \infty]{} f^{(k)} \quad \text{uniformly on } \mathbb{T}.$$

By (2.1), this yields

$$\|f^{[k]} - f_j^{[k]}\|_{L^\infty(\mathbb{T}^{k+1})} = \|(f - f_j)^{[k]}\|_{L^\infty(\mathbb{T}^{k+1})} \leq d_k \|f^{(k)} - f_j^{(k)}\|_{L^\infty(\mathbb{T})} \xrightarrow[j \to \infty]{} 0.$$

Now, let $(\lambda_1, \ldots, \lambda_{n+1}) \in \mathbb{T}^{n+1}$ be outside the diagonal of $\mathbb{T}^{n+1}$. Let $1 \leq i \leq n$ be such that $\lambda_i \neq \lambda_{i+1}$. It follows from the symmetry of $f_j^{[n]}$ that

$$\begin{aligned} &f_j^{[n]}(\lambda_1, \ldots, \lambda_{n+1}) \\ &\quad = \frac{f_j^{[n-1]}(\lambda_1, \ldots, \lambda_i, \lambda_{i+2}, \ldots, \lambda_{n+1}) - f_j^{[n-1]}(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{n+1})}{\lambda_i - \lambda_{i+1}}. \end{aligned}$$

Hence, the pointwise convergence of $(f_j^{[n-1]})_j$ to $f^{[n-1]}$ implies the convergence of $(f_j^{[n]}(\lambda_1, \ldots, \lambda_{n+1}))_j$ to $f^{[n]}(\lambda_1, \ldots, \lambda_{n+1})$. ∎

The next lemma gives the construction of another sequence of functions $(f_j)_j$ whose advantage is that $(f_j^{[n]})_j$ is pointwise convergent to $f^{[n]}$ everywhere. However, it is not clear that we can estimate the derivatives $f_j^{(n)}$ as efficiently as in Lemma 2.1. For that reason, and even if we can have a better estimate, we will only prove that the derivatives are bounded, which is enough for our purpose. This result will be useful in Section 4, as it will allow us to circumvent certain combinatorial and computational difficulties; see Proposition 4.4.

Lemma 2.2. *Let $n \in \mathbb{N}$ and let $f : \mathbb{T} \to \mathbb{C}$ be an n-times differentiable function such that $f^{(n)}$ is bounded. Then there exists a sequence $(f_j)_j \subset C^n(\mathbb{T})$ such that:*

(1) *For every* $1 \le k \le n-1$, *the sequence* $(f_j^{[k]})_j$ *is uniformly convergent to* $f^{[k]}$ *on* $\mathbb{T}^{k+1}$.
(2) *The sequence* $(f_j^{[n]})_j$ *is pointwise convergent to* $f^{[n]}$ *on* $\mathbb{T}^{k+1}$.
(3) *There exists a constant* $M > 0$ *such that, for every* $1 \le k \le n$ *and every* $j \in \mathbb{N}$,

$$\|f_j^{(k)}\|_{L^\infty(\mathbb{T})} \le M.$$

Proof. For a function $g : \mathbb{T} \to \mathbb{C}$, we let $\tilde{g} : \mathbb{R} \to \mathbb{C}$ be the 2π-periodic function defined for every $t \in \mathbb{R}$ by $\tilde{g}(t) = g(e^{it})$. Then g is n-times differentiable on $\mathbb{T}$ if and only if $\tilde{g}$ is n-times differentiable on $\mathbb{R}$. Moreover, by induction (or using Faà di Bruno's formula), we can prove that for every $1 \le k \le n$, there exist constants $a_{1,k}, \dots, a_{k,k}, b_{1,k}, \dots, b_{k,k} \in \mathbb{C}$ independent of g (which we do not need to make explicit) such that, for every $e^{it} \in \mathbb{T}$,

$$\tilde{g}^{(k)}(t) = \sum_{p=1}^{k} a_{p,k} e^{ipt} g^{(p)}(e^{it}) \quad \text{and} \quad g^{(k)}(e^{it}) = e^{-ikt} \sum_{p=1}^{k} b_{p,k} \tilde{g}^{(p)}(t). \tag{2.2}$$

For any $j \in \mathbb{N}$, define $\tilde{f}_j : \mathbb{R} \to \mathbb{C}$ by

$$\forall t \in \mathbb{R}, \quad \tilde{f}_j(t) = j \int_0^t (\tilde{f}(u + 1/j) - \tilde{f}(u))\, du + \tilde{f}(0).$$

Then $\tilde{f}_j$ is 2π-periodic, $\tilde{f}_j \in C^n(\mathbb{R})$ and for every $1 \le k \le n$ and every $t \in \mathbb{R}$,

$$\tilde{f}_j^{(k)}(t) = j\big(\tilde{f}^{(k-1)}(t + 1/j) - \tilde{f}^{(k-1)}(t)\big). \tag{2.3}$$

It is then easy to check that for every $1 \le k \le n-1$,

$$\tilde{f}_j^{(k)} \xrightarrow[j\to\infty]{} \tilde{f}^{(k)} \quad \text{uniformly on } \mathbb{R} \tag{2.4}$$

and

$$\tilde{f}_j^{(n)} \xrightarrow[j\to\infty]{} \tilde{f}^{(n)} \quad \text{pointwise on } \mathbb{R}. \tag{2.5}$$

Moreover, for every $1 \le k \le n$,

$$\forall t \in \mathbb{R}, \quad |\tilde{f}_j^{(k)}(t)| \le \|\tilde{f}^{(k)}\|_{L^\infty(\mathbb{R})}. \tag{2.6}$$

Let us show that the sequence $(f_j)_j$, where $f_j(e^{it}) = \tilde{f}_j(t)$, satisfies conditions (1)–(3). First, f_j is n-times differentiable on $\mathbb{T}$, and according to (2.2) we have, for every $1 \le k \le n$,

$$f_j^{(k)}(e^{it}) = e^{-ikt} \sum_{p=1}^{k} b_{p,k} \tilde{f}_j^{(p)}(t).$$

It follows that $f_j^{(k)}$ is continuous on $\mathbb{T}$ so that $f_j \in C^n(\mathbb{T})$. Moreover, for

$1 \le k \le n-1$ and by (2.4), $(f_j^{(k)})_j$ is uniformly convergent to the function

$$z = e^{it} \mapsto e^{-ikt} \sum_{p=1}^{k} b_{p,k} \tilde{f}^{(p)}(t) = f^{(k)}(e^{it}),$$

and similarly, by (2.4) and (2.5),

$$f_j^{(n)} \xrightarrow[j\to\infty]{} f^{(n)} \quad \text{pointwise on } \mathbb{T}. \tag{2.7}$$

Hence, according to (2.1), we find that for $1 \le k \le n-1$,

$$\|f^{[k]} - f_j^{[k]}\|_{L^\infty(\mathbb{T}^{k+1})} = \|(f - f_j)^{[k]}\|_{L^\infty(\mathbb{T}^{k+1})} \le d_k \|f^{(k)} - f_j^{(k)}\|_{L^\infty(\mathbb{T})} \xrightarrow[j\to\infty]{} 0,$$

which gives (1). As in the proof of Lemma 2.1, it also follows that $(f_j^{[n]})_j$ is pointwise convergent to $f^{[n]}$ outside the diagonal of $\mathbb{T}^{k+1}$, and if $(\lambda, \dots, \lambda) \in \mathbb{T}^{n+1}$ we have, by (2.7),

$$f_j^{[n]}(\lambda, \dots, \lambda) = \frac{1}{n!} f_j^{(n)}(\lambda) \xrightarrow[j\to\infty]{} \frac{1}{n!} f^{(n)}(\lambda) = f^{[n]}(\lambda, \dots, \lambda),$$

which proves that $(f_j)_j$ satisfies (2).

Finally, by (2.6), the sequences $(\tilde{f}_j^{(k)})_j$, $1 \le k \le n$, are uniformly bounded on $\mathbb{R}$, and by (2.2), this implies that the sequences $(f_j^{(k)})_j$, $1 \le k \le n$, are uniformly bounded on $\mathbb{T}$. This yields (3) and finishes the proof of the lemma. ∎

3. Multiple operator integrals. In this section, we first recall the definition of multiple operator integrals as constructed in [8, Section 3]. Other approaches to operator integration require a certain regularity of the symbol, while this construction is more general and thus fits in with this paper's scope. For other approaches, we refer to [27], as well as the references therein. Next, we extend the result on the $\mathcal{S}^p$-boundedness of such mappings when the symbol is a divided difference $f^{[n]}$ for a (non-continuously) n-times differentiable function f with bounded nth derivative. Finally, we prove an important perturbation formula and give some of its consequences, which are key for our analysis.

3.1. Definition and background. Let A be a normal operator on $\mathcal{H}$. In this paper, A will be a unitary operator most of the time, but we will also need the case of selfadjoint operators in Section 4. Denote by E^A the corresponding spectral measure. For any bounded Borel function $f : \sigma(A) \to \mathbb{C}$, one defines an element $f(A) \in \mathcal{B}(\mathcal{H})$ by setting

$$f(A) := \int_{\sigma(A)} f(t)\, dE^A(t).$$

According to [9, Section 15], there exists a finite positive measure λ_A on the Borel subsets of $\sigma(A)$ such that E^A and λ_A have the same sets of measure zero. If $f : \sigma(A) \to \mathbb{C}$ is bounded, then by [9, Theorem 15.10], the operator $f(A)$ only depends on the class of f in $L^\infty(\lambda_A)$ and it induces a w^*-continuous $*$-representation

$$L^\infty(\lambda_A) \ni f \mapsto f(A) \in \mathcal{B}(\mathcal{H}).$$

The measure λ_A is called the *scalar-valued spectral measure* for A.

Let $n \in \mathbb{N}$, and let $A_1, \dots, A_{n+1}$ be normal operators on $\mathcal{H}$ with scalar-valued spectral measures $\lambda_{A_1}, \dots, \lambda_{A_{n+1}}$ Let

$$\Gamma : L^\infty(\lambda_{A_1}) \otimes \cdots \otimes L^\infty(\lambda_{A_{n+1}}) \to \mathcal{B}_n(\mathcal{S}^2(\mathcal{H}))$$

be the linear map such that for any $f_i \in L^\infty(\lambda_{A_i})$, $1 \leq i \leq n+1$, and for any $K_1, \dots, K_n \in \mathcal{S}^2(\mathcal{H})$,

$$\begin{aligned}[]&[\Gamma(f_1 \otimes \cdots \otimes f_{n+1})](K_1, \dots, K_n) \\ &\qquad\qquad = f_1(A_1)K_1 f_2(A_2)K_2 \cdots f_n(A_n)K_n f_{n+1}(A_{n+1}).\end{aligned}$$

The space $L^\infty(\lambda_{A_1}) \otimes \cdots \otimes L^\infty(\lambda_{A_{n+1}})$ is w^*-dense in $L^\infty(\prod_{i=1}^{n+1} \lambda_{A_i})$, and according to [8, Proposition 3.4 and Corollary 3.9], Γ extends to a unique w^*-continuous isometry denoted by

$$\Gamma^{A_1,\dots,A_{n+1}} : L^\infty\Big(\prod_{i=1}^{n+1} \lambda_{A_i}\Big) \to \mathcal{B}_n(\mathcal{S}^2(\mathcal{H})).$$

As recalled in the introduction, $\mathcal{B}_n(\mathcal{S}^2(\mathcal{H}))$ is a dual space, and the w^*-continuity of $\Gamma^{A_1,\dots,A_{n+1}}$ means that if a net $(\varphi_i)_{i\in I}$ in $L^\infty(\prod_{i=1}^{n+1} \lambda_{A_i})$ converges to $\varphi \in L^\infty(\prod_{i=1}^{n+1} \lambda_{A_i})$ in the w^*-topology, then for any $K_1, \dots, K_n \in \mathcal{S}^2(\mathcal{H})$, the net

$$([\Gamma^{A_1,\dots,A_{n+1}}(\varphi_i)](K_1, \dots, K_n))_{i\in I}$$

converges to $[\Gamma^{A_1,\dots,A_{n+1}}(\varphi)](K_1, \dots, K_n)$ weakly in $\mathcal{S}^2(\mathcal{H})$.

Definition 3.1. For $\varphi \in L^\infty(\prod_{i=1}^{n+1} \lambda_{A_i})$, the transformation

$$\Gamma^{A_1,\dots,A_{n+1}}(\varphi)$$

is called the *multiple operator integral* associated to $A_1, \dots, A_{n+1}$ and φ. The element φ is sometimes referred to as the *symbol* of the multiple operator integral.

To conclude this subsection, note that one can define

$$\Gamma^{A_1,\dots,A_{n+1}}(\varphi) : \mathcal{S}^2(\mathcal{H}) \times \cdots \times \mathcal{S}^2(\mathcal{H}) \to \mathcal{S}^2(\mathcal{H})$$

for any bounded Borel function $\varphi : \mathcal{U} \to \mathbb{C}$ such that $\prod_{i=1}^{n+1} \sigma(A_i) \subset \mathcal{U}$ by setting

$$\Gamma^{A_1,\dots,A_{n+1}}(\varphi) := \Gamma^{A_1,\dots,A_{n+1}}(\widetilde{\varphi}),$$

where $\widetilde{\varphi}$ is the class of the restriction $\varphi|_{\sigma(A_1)\times\cdots\times\sigma(A_{n+1})}$ in $L^\infty(\prod_{i=1}^{n+1} \lambda_{A_i})$.

3.2. $\mathcal{S}^p$-boundedness and perturbation estimate. Let $1<p<\infty$, $n \in \mathbb{N}$, and let $U_1,\dots,U_{n+1}$ be unitaries on $\mathcal{H}$. In this subsection, we first establish that for every n-times differentiable function f on $\mathbb{T}$ with bounded nth derivative, for the symbol $\varphi = f^{[n]}$, we have $\Gamma^{U_1,\dots,U_{n+1}}(f^{[n]}) \in \mathcal{B}_n(\mathcal{S}^p(\mathcal{H}))$.

More precisely, and more generally, we will show the following. If $1 < p, p_j < \infty$, $j = 1,\dots,n$ are such that $1/p = 1/p_1 + \cdots + 1/p_n$ and $\mathcal{S}^2(\mathcal{H}) \cap \mathcal{S}^{p_j}(\mathcal{H})$ is equipped with the $\|\cdot\|_{p_j}$-norm, the n-linear mapping

$$\Gamma^{U_1,\dots,U_{n+1}}(f^{[n]}) : (\mathcal{S}^2(\mathcal{H}) \cap \mathcal{S}^{p_1}(\mathcal{H})) \times \cdots \times (\mathcal{S}^2(\mathcal{H}) \cap \mathcal{S}^{p_n}(\mathcal{H})) \to \mathcal{S}^p(\mathcal{H})$$

is bounded. In particular, by density, it uniquely extends to an element

$$\Gamma^{U_1,\dots,U_{n+1}}(f^{[n]}) \in \mathcal{B}_n(\mathcal{S}^{p_1}(\mathcal{H}) \times \cdots \times \mathcal{S}^{p_n}(\mathcal{H}), \mathcal{S}^p(\mathcal{H})).$$

This result has been established for $n=1$ and a Lipschitz function f on $\mathbb{T}$ in [2, Theorem 2], and in [5, Theorem 2.3] for general $n \in \mathbb{N}$ and f with continuous nth derivative $f^{(n)}$. The selfadjoint counterpart of this result, that is, for an n-times differentable function $g : \mathbb{R} \to \mathbb{C}$ with bounded derivatives $g', \dots, g^{(n)}$, has been proved in [6, Theorem 2.7]. We will need this fact in Section 4 when considering functions of selfadjoint operators.

Let us start with the following lemma which is the unitary analogue of [6, Lemma 2.3]. It will be used throughout this paper. Note that it holds true even for normal operators, with the same proof.

Lemma 3.2. *Let $n \in \mathbb{N}$ and let $p_1,\dots,p_n,p \in (1,\infty)$ be such that $1/p = 1/p_1 + \cdots + 1/p_n$. Let $U_1,\dots,U_{n+1}$ be unitary operators on $\mathcal{H}$. Let $(\varphi_k)_{k\geq 1}$, $\varphi \in L^\infty(\prod_{i=1}^{n+1} \lambda_{U_i})$ and assume that $(\varphi_k)_k$ is w^*-convergent to φ and that there exists $C \geq 0$ such that, for every $k \geq 1$,*

$$\|\Gamma^{U_1,\dots,U_{n+1}}(\varphi_k)\|_{\mathcal{B}_n(\mathcal{S}^{p_1}\times\cdots\times\mathcal{S}^{p_n},\mathcal{S}^p)} \leq C.$$

Then $\Gamma^{U_1,\dots,U_{n+1}}(\varphi) \in \mathcal{B}_n(\mathcal{S}^{p_1} \times \cdots \times \mathcal{S}^{p_n}, \mathcal{S}^p)$ and

$$\|\Gamma^{U_1,\dots,U_{n+1}}(\varphi)\|_{\mathcal{B}_n(\mathcal{S}^{p_1}\times\cdots\times\mathcal{S}^{p_n},\mathcal{S}^p)} \leq C.$$

Moreover, for any $X_i \in \mathcal{S}^{p_i}(\mathcal{H}), 1 \leq i \leq n$,

$$[\Gamma^{U_1,\dots,U_{n+1}}(\varphi_k)](X_1,\dots,X_n) \xrightarrow[k\to\infty]{} [\Gamma^{U_1,\dots,U_{n+1}}(\varphi)](X_1,\dots,X_n)$$

weakly in $\mathcal{S}^p(\mathcal{H})$.

The following states that [5, Theorem 2.3] remains true when we drop the assumption of continuity of $f^{(n)}$. It is crucial because it ensures that all the operators that will appear in the rest of the paper belong to $\mathcal{S}^p(\mathcal{H})$.

Theorem 3.3. *Let $n \in \mathbb{N}$ and let $f : \mathbb{T} \to \mathbb{C}$ be n-times differentiable such that $f^{(n)}$ is bounded. Let $1 < p, p_j < \infty$, $j = 1,\dots,n$, be such that*

$1/p = 1/p_1 + \cdots + 1/p_n$. *Let* $U_1, \ldots, U_{n+1}$ *be unitary operators on* $\mathcal{H}$. *Then*

$$\Gamma^{U_1,\ldots,U_{n+1}}(f^{[n]}) \in \mathcal{B}_n(\mathcal{S}^{p_1}(\mathcal{H}) \times \cdots \times \mathcal{S}^{p_n}(\mathcal{H}), \mathcal{S}^p(\mathcal{H}))$$

and there exists $C_{p,n} > 0$ *depending only on* $n, p_1, \ldots, p_n$ *such that*

$$\|\Gamma^{U_1,\ldots,U_{n+1}}(f^{[n]})\|_{\mathcal{B}_n(\mathcal{S}^{p_1}(\mathcal{H})\times\cdots\times\mathcal{S}^{p_n}(\mathcal{H}),\mathcal{S}^p(\mathcal{H}))} \leq C_{p,n}\|f^{(n)}\|_{L^\infty(\mathbb{T})}. \tag{3.1}$$

In particular, $\Gamma^{U_1,\ldots,U_{n+1}}(f^{[n]}) \in \mathcal{B}_n(\mathcal{S}^p(\mathcal{H}))$, *with*

$$\|\Gamma^{U_1,\ldots,U_{n+1}}(f^{[n]})\|_{\mathcal{B}_n(\mathcal{S}^p)} \leq C_{p,n}\|f^{(n)}\|_{L^\infty(\mathbb{T})}. \tag{3.2}$$

To prove this theorem, we will need the following lemma. It is certainly well-known to specialists but we include a proof for the convenience of the reader.

LEMMA 3.4. *Let* $n \in \mathbb{N}$ *and let* $f : \mathbb{T} \to \mathbb{C}$ *be* n*-times differentiable such that* $f^{(n)}$ *is bounded. Let* $1 < p, p_j < \infty$, $j = 1, \ldots, n$, *be such that* $1/p = 1/p_1 + \cdots + 1/p_n$. *Let* $U_1, \ldots, U_{n+1}$ *be unitary operators on* $\mathcal{H}$. *Let* $\Delta := \{(\lambda_1, \ldots, \lambda_{n+1}) \mid \lambda_1 = \cdots = \lambda_{n+1}\}$ *be the diagonal of* $\mathbb{T}^{n+1}$. *Then*

$$\Gamma^{U_1,\ldots,U_{n+1}}(f^{[n]}\chi_\Delta) \in \mathcal{B}_n(\mathcal{S}^{p_1}(\mathcal{H}) \times \cdots \times \mathcal{S}^{p_n}(\mathcal{H}), \mathcal{S}^p(\mathcal{H}))$$

and

$$\|\Gamma^{U_1,\ldots,U_{n+1}}(f^{[n]}\chi_\Delta)\| \leq \frac{1}{n!}\|f^{(n)}\|_{L^\infty(\mathbb{T})}.$$

Proof. Let $(g_k)_k$ be a sequence of continuous functions converging pointwise to $f^{(n)}$ on $\mathbb{T}$ and such that for every k, $\|g_k\|_{L^\infty(\mathbb{T})} \leq \|f^{(n)}\|_{L^\infty(\mathbb{T})}$ (take e.g. $g_k(z) = \frac{k(f^{(n-1)}(ze^{i/k}) - f^{(n-1)}(z))}{iz}$). Let $\tilde{g}_k$ be defined, for any $(\lambda_1, \ldots, \lambda_{n+1})$ in $\mathbb{T}^{n+1}$, by

$$\tilde{g}_k(\lambda_1, \ldots, \lambda_{n+1}) = \frac{1}{n!}g_k(\lambda_1)\chi_\Delta(\lambda_1, \ldots, \lambda_{n+1}).$$

Note that

$$(f^{[n]}\chi_\Delta)(\lambda_1, \ldots, \lambda_{n+1}) = \frac{1}{n!}f^{(n)}(\lambda_1)\chi_\Delta(\lambda_1, \ldots, \lambda_{n+1}).$$

Hence, by Lebesgue's dominated convergence theorem, $(\tilde{g}_k)_k$ w^*-converges to $f^{[n]}\chi_\Delta$ in $L^\infty(\lambda_{U_1} \times \cdots \times \lambda_{U_{n+1}})$. In particular, to prove the lemma, it suffices, according to Lemma 3.2, to prove that $\Gamma^{U_1,\ldots,U_{n+1}}(\tilde{g}_k) \in \mathcal{B}_n(\mathcal{S}^{p_1}(\mathcal{H}) \times \cdots \times \mathcal{S}^{p_n}(\mathcal{H}), \mathcal{S}^p(\mathcal{H}))$ with norm less than $\frac{1}{n!}\|f^{(n)}\|_{L^\infty(\mathbb{T})}$. To simplify the notations, set $h := \frac{1}{n!}g_r$ and $\tilde{h} := \tilde{g}_r$ for some fixed $r \in \mathbb{N}$.

Let $m \in \mathbb{N}$. Let $A_{m,k} := \{e^{2i\pi t} \mid k/2^m \leq t < (k+1)/2^m\}$ and define $P^j_{m,k} := E^{U_j}(A_{m,k})$. Then $\sum_{k=0}^{2^m-1} P^j_{m,k} = I_\mathcal{H}$ for every $1 \leq j \leq n+1$. Let $2 \leq q \leq n$ and let $K \in \mathcal{S}^{p_q}(\mathcal{H})$. Define

$$U_m(t) = \sum_{k=0}^{2^m-1} e^{ikt}P^q_{m,k} \quad \text{and} \quad V_m(t) = \sum_{k=0}^{2^m-1} e^{-ikt}P^{q+1}_{m,k}.$$

For every $t \in \mathbb{R}$, $U_m(t)$ and $V_m(t)$ are unitaries on $\mathcal{H}$ and we have

$$U_m(t)KV_m(t) = \sum_{k,l=0}^{2^m-1} e^{i(k-l)t} P^q_{m,k} K P^{q+1}_{m,l},$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi} U_m(t)KV_m(t)\,dt = \sum_{k=0}^{2^m-1} P^q_{m,k} K P^{q+1}_{m,k},$$

which in turn yields

$$\Big\| \sum_{k=0}^{2^m-1} P^q_{m,k} K P^{q+1}_{m,k} \Big\|_{p_q} \leq \frac{1}{2\pi}\int_0^{2\pi} \|U_m(t)KV_m(t)\|_{p_q}\,dt = \|K\|_{p_q}. \tag{3.3}$$

For $q=1$, one defines

$$\widetilde{U}_m(t) := \sum_{k=0}^{2^m-1} h(e^{2i\pi k/2^m}) e^{ikt} P^1_{m,k} \quad \text{and} \quad V_m(t) = \sum_{k=0}^{2^m-1} e^{-ikt} P^2_{m,k}.$$

Then $\|\widetilde{U}_m(t)\| \leq \|h\|_{L^\infty(\mathbb{T})}$ and proceeding as above, we get

$$\begin{aligned}\Big\| \sum_{k=0}^{2^m-1} h(e^{2i\pi k/2^m}) P^1_{m,k} K P^2_{m,k} \Big\|_{p_1} & \\ \leq \|h\|_{L^\infty(\mathbb{T})} \|K\|_{p_1} &\leq \frac{1}{n!}\|f^{(n)}\|_{L^\infty(\mathbb{T})}\|K\|_{p_1}.\end{aligned} \tag{3.4}$$

Next, let $E_{m,k} = \prod_{i=1}^{n+1} A_{m,k}$ be the Cartesian product of $n+1$ copies of $A_{m,k}$. Define

$$\varphi_m := \sum_{k=0}^{2^m-1} h(e^{2i\pi k/2^m}) \chi_{E_{m,k}}.$$

For every $1 \leq i \leq n$, let $K_i \in \mathcal{S}^{p_i}(\mathcal{H})$. We have, by definition of multiple operator integrals and by orthogonality,

$$\begin{aligned} &[\Gamma^{U_1,\ldots,U_{n+1}}(\varphi_m)](K_1,\ldots,K_n) \\ &\qquad = \sum_{k=0}^{2^m-1} h(e^{2i\pi k/2^m}) P^1_{m,k} K_1 P^2_{m,k} \cdots P^n_{m,k} K_n P^{n+1}_{m,k} \\ &\qquad = \Big(\sum_{k=0}^{2^m-1} h(e^{2i\pi k/2^m}) P^1_{m,k} K_1 P^2_{m,k} \Big) \\ &\qquad\quad \cdot \Big(\sum_{k=0}^{2^m-1} P^2_{m,k} K_2 P^3_{m,k} \Big) \cdots \Big(\sum_{k=0}^{2^m-1} P^n_{m,k} K_n P^{n+1}_{m,k} \Big). \end{aligned}$$

It follows from (3.3) and (3.4) that

$$\|[\Gamma^{U_1,\dots,U_{n+1}}(\varphi_m)](K_1,\dots,K_n)\|_p \leq \frac{1}{n!}\|f^{(n)}\|_{L^\infty(\mathbb{T})}\|K_1\|_{p_1}\cdots\|K_n\|_{p_n}.$$

Next, we show that $\varphi_m \xrightarrow[m\to\infty]{} \tilde{h}$ pointwise on $\mathbb{T}^{n+1}$. First, let $(\lambda,\dots,\lambda) \in \mathbb{T}^{n+1}$ be on the diagonal of $\mathbb{T}^{n+1}$. Write $\lambda = e^{2i\pi t}$ where $0 \leq t < 1$, and for every $m \in \mathbb{N}$ let k_m be the unique integer such that $0 \leq k_m \leq 2^m - 1$ and $\lambda \in A_{m,k_m}$. Then, for every m, we have $\frac{k_m}{2^m} \leq t < \frac{k_m+1}{2^m}$, which implies that $\lim_{m\to\infty}\frac{k_m}{2^m} = t$. In particular, by continuity of h,

$$\varphi_m(\lambda,\dots,\lambda) = h(e^{2i\pi k_m/2^m}) \xrightarrow[m\to\infty]{} h(e^{2i\pi t}) = h(\lambda) = \tilde{h}(\lambda,\dots,\lambda).$$

If $(\lambda_1,\dots,\lambda_{n+1}) \notin \Delta$, then, for every m large enough and for every $0 \leq k \leq 2^m - 1$, $(\lambda_1,\dots,\lambda_{n+1}) \notin E_{m,k}$, so that

$$\varphi_m(\lambda_1,\dots,\lambda_{n+1}) = 0 = \tilde{h}(\lambda_1,\dots,\lambda_{n+1}).$$

To conclude the proof, notice that $\|\varphi_m\|_{L^\infty(\mathbb{T}^{n+1})} \leq \|\tilde{h}\|_{L^\infty(\mathbb{T}^{n+1})}$, hence, by Lebesgue's dominated convergence theorem, $(\varphi_m)_m$ w^*-converges to $\tilde{h} = \tilde{g}_r$ in $L^\infty(\lambda_{U_1}\times\cdots\times\lambda_{U_{n+1}})$. Using Lemma 3.2, we get

$$\|\Gamma^{U_1,\dots,U_{n+1}}(\tilde{g}_r)\| \leq \frac{1}{n!}\|f^{(n)}\|_{L^\infty(\mathbb{T})},$$

and the conclusion of the lemma follows. ■

We now turn to the proof of Theorem 3.3.

Proof of Theorem 3.3. Let $\Delta := \{(\lambda_1,\dots,\lambda_{n+1}) \mid \lambda_1 = \cdots = \lambda_{n+1}\}$. By Lemma 3.4,

$$\Gamma^{U_1,\dots,U_{n+1}}(f^{[n]}\chi_\Delta) \in \mathcal{B}_n(\mathcal{S}^{p_1}(\mathcal{H})\times\cdots\times\mathcal{S}^{p_n}(\mathcal{H}),\mathcal{S}^p(\mathcal{H}))$$

with norm at most $\frac{1}{n!}\|f^{(n)}\|_{L^\infty(\mathbb{T})}$. Hence, it suffices to show the boundedness of $\Gamma^{U_1,\dots,U_{n+1}}(f^{[n]}(1-\chi_\Delta))$. Let $(f_j)_j$ be the sequence of trigonometric polynomials given by Lemma 2.1. Let $T_j := \Gamma^{U_1,\dots,U_{n+1}}(f_j^{[n]})$ and $\tilde{T}_j := \Gamma^{U_1,\dots,U_{n+1}}(f_j^{[n]}\chi_\Delta)$. According to [5, Theorem 2.3] and Lemma 3.4,

$$T_j, \tilde{T}_j \in \mathcal{B}_n(\mathcal{S}^{p_1}(\mathcal{H})\times\cdots\times\mathcal{S}^{p_n}(\mathcal{H}),\mathcal{S}^p(\mathcal{H})),$$

and there exists a constant $c_{p,n} > 0$ such that

$$\|T_j\| \leq c_{p,n}\|f_j^{(n)}\|_{L^\infty(\mathbb{T})} \leq c_{p,n}\|f^{(n)}\|_{L^\infty(\mathbb{T})},$$
$$\|\tilde{T}_j\| \leq \frac{1}{n!}\|f_j^{(n)}\|_{L^\infty(\mathbb{T})} \leq \frac{1}{n!}\|f^{(n)}\|_{L^\infty(\mathbb{T})}.$$

Notice that $1 - \chi_\Delta = \chi_\Omega$ where $\Omega := \mathbb{T}^{n+1} \setminus \Delta$, so that, according to Lemma 2.1, $f_j^{[n]}(1-\chi_\Delta)$ is pointwise convergent to $f^{[n]}(1-\chi_\Delta)$ and

$$\|f_j^{[n]}(1-\chi_\Delta)\|_{L^\infty(\mathbb{T}^{n+1})} \leq \|f_j^{[n]}\|_{L^\infty(\mathbb{T}^{n+1})} \leq d_n\|f^{(n)}\|_{L^\infty(\mathbb{T})}.$$

Hence, $(f_j^{[n]}(1-\chi_\Delta))_j$ w^*-converges to $f^{[n]}(1-\chi_\Delta)$ in $L^\infty(\lambda_{U_1}\times\cdots\times\lambda_{U_{n+1}})$. Since

$$\|\Gamma^{U_1,\ldots,U_{n+1}}(f_j^{[n]}(1-\chi_\Delta))\| = \|T_j - \tilde{T}_j\| \leq \left(c_{p,n} + \frac{1}{n!}\right)\|f^{(n)}\|_{L^\infty(\mathbb{T})},$$

it follows from Lemma 3.2 that $\Gamma^{U_1,\ldots,U_{n+1}}(f^{[n]}(1-\chi_\Delta)) \in \mathcal{B}_n(\mathcal{S}^{p_1}(\mathcal{H})\times\cdots\times\mathcal{S}^{p_n}(\mathcal{H}),\mathcal{S}^p(\mathcal{H}))$ and

$$\|\Gamma^{U_1,\ldots,U_{n+1}}(f^{[n]}(1-\chi_\Delta))\| \leq \left(c_{p,n} + \frac{1}{n!}\right)\|f^{(n)}\|_{L^\infty(\mathbb{T})}.$$

This concludes the proof of the theorem. ∎

The next proposition is a crucial perturbation formula. It is key whenever the differentiability of operator functions with $\mathcal{S}^p$-perturbation is studied. In the unitary setting, it generalizes [5, Proposition 3.5] where the result was proved for $f \in C^n(\mathbb{T})$.

Proposition 3.5. *Let $1 < p < \infty$ and $n \in \mathbb{N}_{\geq 2}$. Let $U_1,\ldots,U_{n-1}$, $U, V \in \mathcal{U}(\mathcal{H})$ be such that $U - V \in \mathcal{S}^p(\mathcal{H})$. Let $f : \mathbb{T} \to \mathbb{C}$ be n-times differentiable on $\mathbb{T}$ such that $f^{(n)}$ is bounded. Then, for all $K_1,\ldots,K_{n-1} \in \mathcal{S}^p(\mathcal{H})$ and for any $1 \leq i \leq n$ we have*

$$\begin{aligned}&[\Gamma^{U_1,\ldots,U_{i-1},U,U_i,\ldots,U_{n-1}}(f^{[n-1]}) - \Gamma^{U_1,\ldots,U_{i-1},V,U_i,\ldots,U_{n-1}}(f^{[n-1]})](K_1,\ldots,K_{n-1})\\ &\qquad= [\Gamma^{U_1,\ldots,U_{i-1},U,V,U_i,\ldots,U_{n-1}}(f^{[n]})](K_1,\ldots,K_{i-1},U-V,K_i,\ldots,K_{n-1}).\end{aligned}$$

Proof. Fix $1 \leq i \leq n$ and let $(f_j)_j \subset C^n(\mathbb{T})$ be given by Lemma 2.2. By [5, Proposition 2.5], we have

(3.5)
$$\begin{aligned}&[\Gamma^{U_1,\ldots,U_{i-1},U,U_i,\ldots,U_{n-1}}(f_j^{[n-1]}) - \Gamma^{U_1,\ldots,U_{i-1},V,U_i,\ldots,U_{n-1}}(f_j^{[n-1]})](K_1,\ldots,K_{n-1})\\ &\qquad= [\Gamma^{U_1,\ldots,U_{i-1},U,V,U_i,\ldots,U_{n-1}}(f_j^{[n]})](K_1,\ldots,K_{i-1},U-V,K_i,\ldots,K_{n-1}).\end{aligned}$$

The sequence $(f_j^{(n-1)})_j$ is uniformly convergent to $f^{(n-1)}$ on $\mathbb{T}$. It follows from Theorem 3.3 that

$$\Gamma^{U_1,\ldots,U_{i-1},U,U_i,\ldots,U_{n-1}}(f_j^{[n-1]}) \xrightarrow[j\to\infty]{} \Gamma^{U_1,\ldots,U_{i-1},U,U_i,\ldots,U_{n-1}}(f^{[n-1]}),$$
$$\Gamma^{U_1,\ldots,U_{i-1},V,U_i,\ldots,U_{n-1}}(f_j^{[n-1]}) \xrightarrow[j\to\infty]{} \Gamma^{U_1,\ldots,U_{i-1},V,U_i,\ldots,U_{n-1}}(f^{[n-1]})$$

in $\mathcal{B}_n(\mathcal{S}^p(\mathcal{H}))$. In particular, they converge pointwise, that is,

$$\begin{aligned}&[\Gamma^{U_1,\ldots,U_{i-1},U,U_i,\ldots,U_{n-1}}(f_j^{[n-1]}) - \Gamma^{U_1,\ldots,U_{i-1},V,U_i,\ldots,U_{n-1}}(f_j^{[n-1]})](K_1,\ldots,K_{n-1})\\ &\xrightarrow[j\to\infty]{}\\ &[\Gamma^{U_1,\ldots,U_{i-1},U,U_i,\ldots,U_{n-1}}(f^{[n-1]}) - \Gamma^{U_1,\ldots,U_{i-1},V,U_i,\ldots,U_{n-1}}(f^{[n-1]})](K_1,\ldots,K_{n-1})\end{aligned}$$

in $\mathcal{S}^p(\mathcal{H})$. Next, by Lemma 2.2 and Lebesgue's dominated convergence theorem, the sequence $(f_j^{[n]})_j$ w^*-converges to $f^{[n]}$ in $L^\infty(\lambda_{U_1} \times \cdots \times \lambda_{U_{i-1}} \times \lambda_U \times \lambda_V \times \lambda_{U_i} \times \cdots \times \lambda_{U_{n+1}})$. It follows from Lemma 3.2 that

$$[\Gamma^{U_1,\ldots,U_{i-1},U,V,U_i,\ldots,U_{n-1}}(f_j^{[n]})](K_1,\ldots,K_{i-1},U-V,K_i,\ldots,K_{n-1})$$

converges weakly (in $\mathcal{S}^p(\mathcal{H})$) to

$$[\Gamma^{U_1,\ldots,U_{i-1},U,V,U_i,\ldots,U_{n-1}}(f^{[n]})](K_1,\ldots,K_{i-1},U-V,K_i,\ldots,K_{n-1}).$$

Hence, taking the limit as $j \to \infty$ in (3.5) in the weak topology of $\mathcal{S}^p(\mathcal{H})$ yields the desired identity. ∎

Corollary 3.6. *Let $1 < p < \infty$ and let $n \in \mathbb{N}_{\geq 2}$. Let $U, V \in \mathcal{U}(\mathcal{H})$ be such that $U - V \in \mathcal{S}^p(\mathcal{H})$. Let $f : \mathbb{T} \to \mathbb{C}$ be n-times differentiable on $\mathbb{T}$ such that $f^{(n)}$ is bounded. Then, for all $K_1, \ldots, K_{n-1} \in \mathcal{S}^p(\mathcal{H})$,*

$$\begin{aligned}&[\Gamma^{(U)^n}(f^{[n-1]}) - \Gamma^{(V)^n}(f^{[n-1]})](K_1,\ldots,K_{n-1})\\ &\qquad= \sum_{i=1}^n [\Gamma^{(U)^i,(V)^{n-i+1}}(f^{[n]})](K_1,\ldots,K_{i-1},U-V,K_i,\ldots,K_{n-1}).\end{aligned}$$

Proof. It suffices to write

$$\begin{aligned}&[\Gamma^{(U)^n}(f^{[n-1]}) - \Gamma^{(V)^n}(f^{[n-1]})](K_1,\ldots,K_{n-1})\\ &\quad= \sum_{i=1}^n [\Gamma^{(U)^i,(V)^{n-i}}(f^{[n-1]}) - \Gamma^{(U)^{i-1},(V)^{n-i+1}}(f^{[n-1]})](K_1,\ldots,K_{n-1})\end{aligned}$$

and then apply Proposition 3.5. ∎

Corollary 3.7. *Let $1 < p < \infty$ and let $n \in \mathbb{N}_{\geq 2}$. Let $U_1, \ldots, U_n, V_1, \ldots, V_n \in \mathcal{U}(\mathcal{H})$ be such that $U_i - V_i \in \mathcal{S}^p(\mathcal{H})$ for every $1 \leq i \leq n$. Let $f : \mathbb{T} \to \mathbb{C}$ be n-times differentiable on $\mathbb{T}$ such that $f^{(n)}$ is bounded. Then there exists $D_{p,n} > 0$ such that*

$$\begin{aligned}&\|[\Gamma^{U_1,\ldots,U_n}(f^{[n-1]}) - \Gamma^{V_1,\ldots,V_n}(f^{[n-1]})]\|_{\mathcal{B}_{n-1}(\mathcal{S}^p(\mathcal{H}))}\\ &\qquad\qquad\leq D_{p,n}\|f^{(n)}\|_{L^\infty(\mathbb{T})} \max_{1\leq k\leq n} \|U_k - V_k\|_p.\end{aligned}$$

Proof. Let $K_1, \ldots, K_{n-1} \in \mathcal{S}^p(\mathcal{H})$. By Proposition 3.5, we have

$$\begin{aligned}&[\Gamma^{U_1,\ldots,U_n}(f^{[n-1]}) - \Gamma^{V_1,\ldots,V_n}(f^{[n-1]})](K_1,\ldots,K_{n-1})\\ &= \sum_{k=1}^n [\Gamma^{U_1,\ldots,U_k,V_{k+1},\ldots,V_n}(f^{[n-1]}) - \Gamma^{U_1,\ldots,U_{k-1},V_k,\ldots,V_n}(f^{[n-1]})](K_1,\ldots,K_{n-1})\\ &= \sum_{k=1}^n [\Gamma^{U_1,\ldots,U_{n-k},V_{n-k},\ldots,V_n}(f^{[n]})](K_1,\ldots,K_{k-1},U_k-V_k,K_k,\ldots,K_{n-1}).\end{aligned}$$

By Theorem 3.3, there exists a constant $C_{p,n}$ such that

$$\begin{aligned}
&\|[\Gamma^{U_1,\ldots,U_n}(f^{[n-1]}) - \Gamma^{V_1,\ldots,V_n}(f^{[n-1]})](K_1,\ldots,K_{n-1})\|_p \\
&\quad \le n \max_{1\le k\le n} \|[\Gamma^{U_1,\ldots,U_{n-k},V_{n-k},\ldots,V_n}(f^{[n]})](K_1,\ldots,K_{k-1},U_k-V_k,K_k,\ldots,K_{n-1})\|_p \\
&\quad \le nC_{p,n}\|f^{(n)}\|_{L^\infty(\mathbb{T})} \max_{1\le k\le n} \|U_k-V_k\|_p\|K_1\|_p\cdots\|K_{n-1}\|_p.
\end{aligned}$$

This concludes the proof of the corollary. ∎

Remark 3.8. Proposition 3.5 and Corollary 3.7 also hold true for $n = 1$. For the perturbation formula, this means that if $U, V \in \mathcal{U}(\mathcal{H})$ are such that $U - V \in \mathcal{S}^p(\mathcal{H})$ and $f : \mathbb{T} \to \mathbb{C}$ is differentiable with bounded f', then

$$f(U) - f(V) = [\Gamma^{U,V}(f^{[1]})](U - V).$$

We refer e.g. to [4]. Alternatively, for a more recent reference, we can first use [5, Proposition 2.5] for $f \in C^1(\mathbb{T})$ (the proof works verbatim for $n = 1$), and then some minor modifications to the proof of Proposition 3.5 will give the desired formula. The bound given in Corollary 3.7 for $n = 1$ simply corresponds to [2, Theorem 2].

4. From selfadjoint to unitary operators. In this section, we will prove a general result on the differentiability of operator functions in the selfadjoint case and then deduce its unitary counterpart using a Cayley transform. This will be the first step towards our main theorem in Section 5.

Let $\mathcal{A} : \mathbb{R} \to \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$ be such that $\mathbb{R} \ni t \mapsto \mathcal{A}(t) - \mathcal{A}(0) \in \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$ is n-times differentiable, and let f be an n-times differentiable function on $\mathbb{R}$. We will show that the function

$$\psi : \mathbb{R} \ni t \mapsto f(\mathcal{A}(t)) - f(\mathcal{A}(0)) \in \mathcal{S}^p(\mathcal{H})$$

is n-times differentiable as well. The particular case $\mathcal{A}(t) = A + tK$, where $A \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$ and $K \in \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$, is the main result of [6]. We will outline the minor changes to make in the proof of [6, Theorem 3.1] as well as in the results therein to obtain our general result in Corollary 4.2.

Let us start with the following, which is the key step prior to a combinatorial reasoning.

Theorem 4.1. *Let $1 < p < \infty$, and let $\mathcal{A} : \mathbb{R} \to \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$ be such that $\tilde{\mathcal{A}} : \mathbb{R} \ni t \mapsto \mathcal{A}(t) - \mathcal{A}(0) \in \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$ is differentiable in a neighborhood I of 0. Let $n \in \mathbb{N}_{\ge 2}$. For every $1 \le i \le n-1$, let $S_i : \mathbb{R} \to \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$ be differentiable on $\mathbb{R}$. Let f be n-times differentiable on $\mathbb{R}$ such that $f^{(n)}$ is bounded and consider the function*

$$\psi : \mathbb{R} \ni t \mapsto [\Gamma^{(\mathcal{A}(t))^n}(f^{[n-1]})](S_1(t),\ldots,S_{n-1}(t)) \in \mathcal{S}^p(\mathcal{H}).$$

Then ψ is differentiable on I and for every $t \in I$,

$$\psi'(t) = \sum_{k=1}^{n-1} [\Gamma^{(\mathcal{A}(t))^n}(f^{[n-1]})](S_1(t), \dots, S_{k-1}(t), S_k'(t), S_{k+1}(t), \dots, S_{n-1}(t)) \\ + \sum_{k=1}^{n} [\Gamma^{(\mathcal{A}(t))^{n+1}}(f^{[n]})](S_1(t), \dots, S_{k-1}(t), \tilde{\mathcal{A}}'(t), S_k(t), \dots, S_{n-1}(t)).$$

Proof. Let us explain the simple modifications to make in [6] to obtain the result. Throughout the proof, we denote $A := \mathcal{A}(0) \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$. Define, for $K, X_1, \dots, X_{n-1} \in \mathcal{S}^p_{\mathrm{sa}}$,

$$\psi_{K,X_1,\dots,X_{n-1}} : \mathbb{R} \ni t \mapsto [\Gamma^{(A+tK)^n}(f^{[n-1]})](X_1, \dots, X_{n-1}).$$

We first want to prove that $\psi_{K,X_1,\dots,X_{n-1}}$ is differentiable at 0. Assume that for every K_0 in a dense subset of $\mathcal{S}^p_{\mathrm{sa}}$, $\psi_{K_0,X_1,\dots,X_{n-1}}$ is differentiable at 0 with

$$\psi'_{K_0,X_1,\dots,X_{n-1}}(0) = \sum_{k=1}^{n} [\Gamma^{(A)^{n+1}}(f^{[n]})](X_1, \dots, X_{k-1}, K_0, X_k, \dots, X_{n-1}).$$

Then, arguing as in the proof of [6, Lemma 3.7], one shows that for every $K \in \mathcal{S}^p_{\mathrm{sa}}$, $\psi_{K,X_1,\dots,X_{n-1}}$ is differentiable at $t = 0$ with the same formula for its derivative. Next, as explained in [15] or in the proof of [6, Theorem 3.1], one can choose

$$\mathcal{F} := \{ i[A, Y] + Z \mid Y, Z \in \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H}) \text{ and } Z \text{ commutes with } A \}$$

as a dense subset of $\mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$. Then we can assume that $K = i[A, Y] + Z \in \mathcal{F}$ and we have to show that $\psi_{K,X_1,\dots,X_{n-1}}$ is differentiable at $t = 0$. The first part of the proof of [6, Theorem 3.1] applies and it tells us that $\psi_{K,X_1,\dots,X_{n-1}}$ has a derivative at 0 if and only if

$$\xi : \mathbb{R} \ni t \mapsto \\ \sum_{k=1}^{n} [\Gamma^{(A+tZ)^{n-k+1},(A)^k}(f^{[n]})](X_1, \dots, X_{n-k}, K, X_{n-k+1}, \dots, X_{n-1})$$

has a limit at 0 (in $\mathcal{S}^p$), and in that case, this limit is $\psi'_{K,X_1,\dots,X_{n-1}}(0)$. But notice that by continuity of multiple operator integrals (the operators $\Gamma^{(A+tZ)^{n-k+1},(A)^k}(f^{[n]})$ are uniformly bounded with respect to $t \in \mathbb{R}$), it is enough to show that this limit exists when $X_1, \dots, X_{n-1}$ are elements of the dense subset $\mathcal{F}$. Hence, one can write $X_i = i[A, Y_i] + Z_i$. The rest of the proof is similar, with obvious modifications, and shows that ξ has indeed a limit at 0 equal to

$$(4.1)\quad \psi'_{K,X_1,\dots,X_{n-1}}(0) = \sum_{k=1}^{n} [\Gamma^{(A)^{n+1}}(f^{[n]})](X_1,\dots,X_{n-k},K,X_{n-k+1},\dots,X_{n-1}),$$

as expected.

Now, let us come back to the function ψ. It is sufficient to prove the formula for $t=0$. Let $K := \mathcal{A}'(0)$. By a straightforward modification of [6, Lemma 3.8], ψ is differentiable at 0 if and only if

$$\tilde{\psi} : \mathbb{R} \ni t \mapsto [\Gamma^{(A+tK)^n}(f^{[n-1]})](S_1(t),\dots,S_{n-1}(t)) \in \mathcal{S}^p(\mathcal{H})$$

is differentiable at 0, and in that case $\psi'(0) = \tilde{\psi}'(0)$. Let us write, for every $1 \le i \le n-1$,

$$S_i(t) = S_i(0) + tS_i'(0) + o_i(t)$$

where $o_i(t) = o(t)$ depends on i. By uniform boundedness of $\Gamma^{(A+tK)^n}(f^{[n-1]})$ for $t \in \mathbb{R}$, and by the multilinearity of operator integrals, we can write

$$\begin{aligned}\tilde{\psi}(t) &= [\Gamma^{(A+tK)^n}(f^{[n-1]})](S_1(0),\dots,S_{n-1}(0)) \\ &+ t\sum_{k=1}^{n-1} [\Gamma^{(A+tK)^n}(f^{[n-1]})](S_1(0),\dots,S_{k-1}(0),S_k'(0),S_{k+1}(0),\dots,S_{n-1}(0)) \\ &+ o(t).\end{aligned}$$

By the first part of the proof, $\tilde{\psi}$ is differentiable at 0, and by (4.1), we get the desired formula. ∎

COROLLARY 4.2. *Let $1 < p < \infty$ and let $n \in \mathbb{N}$. Let $\mathcal{A} : \mathbb{R} \to \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$ be such that $\tilde{\mathcal{A}} : \mathbb{R} \ni t \mapsto \mathcal{A}(t) - \mathcal{A}(0) \in \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$ is n-times differentiable in a neighborhood I of 0. Let f be n-times differentiable on $\mathbb{R}$ such that $f^{(n)}$ is bounded. Then the function*

$$\psi : \mathbb{R} \ni t \mapsto f(\mathcal{A}(t)) - f(\mathcal{A}(0)) \in \mathcal{S}^p(\mathcal{H})$$

is n-times differentiable on I and for every integer $1 \le k \le n$ and every $t \in I$,

$$(4.2)\quad \psi^{(k)}(t) = \sum_{m=1}^{k} \sum_{\substack{l_1,\dots,l_m \ge 1 \\ l_1+\dots+l_m = k}} \frac{k!}{l_1!\cdots l_m!} [\Gamma^{(\mathcal{A}(t))^{m+1}}(f^{[m]})](\tilde{\mathcal{A}}^{(l_1)}(t),\dots,\tilde{\mathcal{A}}^{(l_m)}(t)).$$

Proof. We prove the result by induction on n. The case $n=1$ follows from [15, Theorem 7.13]. When $n \ge 2$, using Theorem 4.1 and employing a combinatorial reasoning as in the proof of [27, Theorem 5.3.4] gives the result. We leave the details to the reader. ∎

REMARK 4.3. When $f \in C^n(\mathbb{R})$, the result of Corollary 4.2 can be proved using [17, Theorem 3.3] instead of Theorem 4.1.

The next proposition corresponds to the main result of this paper (see Theorem 5.1) but with an additional assumption on the function $t \mapsto U(t)$ valued in the unitary operators. We will need it to prove the same result in full generality in Section 5. The proof makes use of Corollary 4.2 which is the corresponding result for selfadjoint operators. We will need the Cayley transform to change the function $t \mapsto f(U(t)) - f(U(0))$ into a function $t \mapsto g(\mathcal{A}(t)) - g(\mathcal{A}(0))$ where $\mathcal{A}$ is valued in the set of selfadjoint operators on $\mathcal{H}$. Denote by $\eta : \mathbb{R} \to \mathbb{T} \setminus \{1\}$ the Cayley transform and by η^{-1} its inverse function, defined by

$$\begin{aligned} \eta : \mathbb{R} &\to \mathbb{T} \setminus \{1\}, & \eta^{-1} : \mathbb{T} \setminus \{1\} &\to \mathbb{R}, \\ x &\mapsto \frac{x+i}{x-i}, & z &\mapsto i\frac{z+1}{z-1}. \end{aligned}$$

If $A \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$, then $\eta(A) \in \mathcal{U}(\mathcal{H})$ and $\sigma(\eta(A)) \subset \mathbb{T} \setminus \{1\}$, and conversely, if $U \in \mathcal{U}(\mathcal{H})$ is such that $1 \notin \sigma(U)$, then $\eta^{-1}(U) \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$.

PROPOSITION 4.4. *Let $1 < p < \infty$ and $n \in \mathbb{N}$. Let $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ be such that the function $\tilde{U} : \mathbb{R} \ni t \mapsto U(t) - U(0) \in \mathcal{S}^p(\mathcal{H})$ is n-times differentiable on $\mathbb{R}$ and assume that $1 \notin \sigma(U(0))$. Let $f : \mathbb{T} \to \mathbb{R}$ be n-times differentiable with bounded nth derivative $f^{(n)}$. Consider the operator-valued function*

$$\varphi : t \mapsto f(U(t)) - f(U(0)) \in \mathcal{S}^p(\mathcal{H}).$$

Then φ is n-times differentiable in a neighborhood I of 0 and for every $t \in I$,

$$\begin{aligned} &\varphi^{(n)}(t) \\ &= \sum_{m=1}^{n} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\cdots+l_m=n}} \frac{n!}{l_1!\cdots l_m!} [\Gamma^{(U(t))^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(t), \ldots, \tilde{U}^{(l_m)}(t)). \end{aligned} \tag{4.3}$$

Proof. By continuity of U, $U(t) \to U(0)$ as $t \to 0$ in the operator norm, and since $1 \notin \sigma(U(0))$ and the spectrum is closed, there is an $a > 0$ and a real interval I around 0 such that, for every $t \in I$, $\sigma(U(t)) \subset C_a$ where $C_a := \{z \in \mathbb{T} \mid |z-1| > a\}$. Note that all functions of operators and multiple operator integrals only depend on the values of the associated function on the spectra of the operators. In particular, one can extend η^{-1} from C_a to a C^∞ function on the whole $\mathbb{T}$ if necessary. Let us define, for every $t \in I$, $\mathcal{A}(t) = \eta^{-1}(U(t)) \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$. Note that for every $t \in I$, $\tilde{\mathcal{A}}(t) := \mathcal{A}(t) - \mathcal{A}(0) \in \mathcal{S}^p(\mathcal{H})$. Indeed, this follows either from [2, Theorem 2] or by the straightforward identity

$$\mathcal{A}(t) - \mathcal{A}(0) = -2i(U(t) - I)^{-1}(U(t) - U(0))(U(0) - I)^{-1},$$

which yields

$$\|\mathcal{A}(t) - \mathcal{A}(0)\|_p \leq 2\|(U(t) - I)^{-1}\| \cdot \|U(t) - U(0)\|_p \cdot \|(U(0) - I)^{-1}\| < \infty.$$

Moreover, $\mathcal{A}$ is n-times differentiable on I. This follows either from [5, Theorem 3.5] or simply by using the fact that η^{-1} is a rational function so one can use standard algebraic identities as above. Let $g : \mathbb{R} \ni t \mapsto f(\eta(t))$ and note that

$$\varphi(t) = f(\eta(\eta^{-1}(U(t)))) - f(\eta(\eta^{-1}(U(0)))) = g(\mathcal{A}(t)) - g(\mathcal{A}(0)).$$

The function g is n-times differentiable and since f and η have bounded derivatives, g has bounded derivatives as well. By Corollary 4.2, φ is n-times differentiable on I and for every $t \in I$,

$$\begin{aligned}\varphi^{(n)}(t) &= \sum_{m=1}^{n} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\cdots+l_m=n}} \frac{n!}{l_1! \cdots l_m!} [\Gamma^{(\mathcal{A}(t))^{m+1}}(g^{[m]})](\tilde{\mathcal{A}}^{(l_1)}(t), \ldots, \tilde{\mathcal{A}}^{(l_m)}(t)) \\ &:= \sum_{m=1}^{n} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\cdots+l_m=n}} D_{g,\mathcal{A}}^{m,l_1,\ldots,l_m}(t).\end{aligned}$$

It remains to show that for a fixed $t \in I$,

$$\begin{aligned}\varphi^{(n)}(t) &= \sum_{m=1}^{n} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\cdots+l_m=n}} \frac{n!}{l_1! \cdots l_m!} [\Gamma^{(U(t))^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(t), \ldots, \tilde{U}^{(l_m)}(t)) \\ &:= \sum_{m=1}^{n} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\cdots+l_m=n}} D_{f,U}^{m,l_1,\ldots,l_m}(t).\end{aligned}$$

To prove this, let $(f_j)_j \subset C^n(\mathbb{T})$ be the sequence given by Lemma 2.2. Then, for every $j \in \mathbb{N}$, the function

$$\varphi_j : t \mapsto f_j(U(t)) - f(U(0)) \in \mathcal{S}^p(\mathcal{H})$$

is n-times differentiable on I and

(4.4)

$$\sum_{m=1}^{n} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\cdots+l_m=n}} D_{f_j,U}^{m,l_1,\ldots,l_m}(t) \overset{\text{(a)}}{=} \varphi_j^{(n)}(t) \overset{\text{(b)}}{=} \sum_{m=1}^{n} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\cdots+l_m=n}} D_{f_j\circ\eta,\mathcal{A}}^{m,l_1,\ldots,l_m}(t).$$

Indeed, since $f_j \in C^n(\mathbb{T})$, the equality (a) comes from [5, Theorem 3.5], while (b) follows from the computations performed for f in the first part of the proof.

Fix $1 \leq m \leq n$ and let $l_1, \dots, l_m \geq 1$ be such that $l_1 + \cdots + l_m = n$. The assumptions on $(f_j)_j$ ensure that $(f_j^{[m]})_j$ w^*-converges to $f^{[m]}$ in $L^\infty(\lambda_{U(t)} \times \cdots \times \lambda_{U(t)})$. By Lemma 3.2, it follows that

$$D_{f_j,U}^{m,l_1,\dots,l_m}(t) \xrightarrow[j\to\infty]{} D_{f,U}^{m,l_1,\dots,l_m}(t)$$

weakly in $\mathcal{S}^p(\mathcal{H})$. On the other hand, by [23, Lemma 2.3], we have, for every $(\lambda_1, \dots, \lambda_{m+1}) \in \mathbb{R}^{m+1}$,

$$\begin{aligned}(f_j \circ \eta)^{[m]}(\lambda_1, \dots, \lambda_{m+1}) \\ = \sum_{k=1}^{m} \sum_{1=i_0<\cdots<i_k=m+1} \frac{(-1)^{k+1} i^{m-k+1}}{2^{m-k+1}} f_j^{[k]}(\eta(\lambda_{i_0}), \dots, \eta(\lambda_{i_k})) \\ \times \prod_{j=1}^{k-1} (\eta(\lambda_{i_j}) - 1)^2 \prod_{l\in\{1,\dots,m+1\}\setminus\{i_1,\dots,i_{k-1}\}} (\eta(\lambda_l) - 1).\end{aligned}$$

This holds true as well if f_j is replaced by f, with the same proof. In particular, the pointwise convergence of $f_j^{[k]}$ to $f^{[k]}$ implies the pointwise convergence of $((f_j \circ \eta)^{[m]})_j$ to $(f \circ \eta)^{[m]} = g^{[m]}$. Together with the boundedness of each $(f_j^{[k]})_j$ and hence the boundedness of $((f_j \circ \eta)^{[m]})_j$, we get the w^*-convergence of $((f_j \circ \eta)^{[m]})_j$ to $g^{[m]}$ in $L^\infty(\lambda_{\mathcal{A}(t)} \times \cdots \times \lambda_{\mathcal{A}(t)})$. By Lemma 3.2 and the paragraph preceding it, it follows that

$$D_{f_j\circ\eta,\mathcal{A}}^{m,l_1,\dots,l_m}(t) \xrightarrow[j\to\infty]{} D_{g,U}^{m,l_1,\dots,l_m}(t)$$

weakly in $\mathcal{S}^p(\mathcal{H})$. Finally, after taking the limit as $j \to \infty$ in the weak topology of $\mathcal{S}^p(\mathcal{H})$ in (4.4), we obtain

$$\sum_{m=1}^{n} \sum_{\substack{l_1,\dots,l_m\geq 1\\ l_1+\cdots+l_m=n}} D_{f,U}^{m,l_1,\dots,l_m}(t) = \sum_{m=1}^{n} \sum_{\substack{l_1,\dots,l_m\geq 1\\ l_1+\cdots+l_m=n}} D_{g,\mathcal{A}}^{m,l_1,\dots,l_m}(t),$$

which gives the desired formula for $\varphi^{(n)}(t)$ and concludes the proof. ∎

Remark 4.5. Proposition 4.4 holds true as well if we simply assume that $\sigma(U(0)) \neq \mathbb{T}$. Indeed, by picking $e^{i\theta} \notin \sigma(U(0))$ and changing the function U to $e^{-i\theta}U$ (in that case, $1 \notin \sigma(e^{-i\theta}U(0))$) and f to $h(z) = f(e^{i\theta}z)$ we get $\varphi : \mathbb{R} \ni t \mapsto f(U(t)) - f(U(0)) = h(e^{-i\theta}U(t)) - h(e^{-i\theta}U(0))$ so that φ is differentiable in a neighborhood of 0. Moreover, it is easy to check that $h^{[n]}(\lambda_1, \dots, \lambda_{n+1}) = e^{in\theta} f^{[n]}(e^{i\theta}\lambda_1, \dots, e^{i\theta}\lambda_{n+1})$ and $(e^{-i\theta}\tilde{U})^{(l_1)}(t) = e^{-i\theta}\tilde{U}^{(l_1)}(t)$ so that

$$\varphi^{(k)}(t)$$
$$= \sum_{m=1}^{k} \sum_{\substack{l_1,\dots,l_m\geq 1\\ l_1+\cdots+l_m=k}} \frac{k!}{l_1!\cdots l_m!}[\Gamma^{(e^{-i\theta}U(t))^{m+1}}(h^{[m]})](e^{-i\theta}\tilde{U}^{(l_1)}(t),\dots,e^{-i\theta}\tilde{U}^{(l_m)}(t))$$
$$= \sum_{m=1}^{k} \sum_{\substack{l_1,\dots,l_m\geq 1\\ l_1+\cdots+l_m=k}} \frac{k!}{l_1!\cdots l_m!}[\Gamma^{(e^{-i\theta}U(t))^{m+1}}((f^{[m]})_r)](\tilde{U}^{(l_1)}(t),\dots,\tilde{U}^{(l_m)}(t)),$$

where $(f^{[m]})_r(\lambda_1,\dots,\lambda_{m+1}) = f^{[m]}(e^{i\theta}\lambda_1,\dots,e^{i\theta}\lambda_{m+1})$. It is now easy to check (using the construction of multiple operator integrals) that

$$\Gamma^{(e^{-i\theta}U(t))^{m+1}}((f^{[m]})_r) = \Gamma^{(U(t))^{m+1}}(f^{[m]}).$$

5. $\mathcal{S}^p$-differentiability for non-continuously differentiable functions. In this section, we will prove the following main result of this paper.

Main Theorem 5.1. *Let $1 < p < \infty$ and $n \in \mathbb{N}$. Let $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ be such that the function $\tilde{U} : \mathbb{R} \ni t \mapsto U(t) - U(0) \in \mathcal{S}^p(\mathcal{H})$ is n-times differentiable on $\mathbb{R}$. Let $f : \mathbb{T} \to \mathbb{R}$ be n-times differentiable with bounded nth derivative $f^{(n)}$. Consider the operator-valued function*

$$\varphi : t \mapsto f(U(t)) - f(U(0)) \in \mathcal{S}^p(\mathcal{H}).$$

Then φ is n-times differentiable on $\mathbb{R}$ and for every integer $1 \leq k \leq n$ and every $t \in \mathbb{R}$,

(5.1)
$$\varphi^{(k)}(t) = \sum_{m=1}^{k} \sum_{\substack{l_1,\dots,l_m\geq 1\\ l_1+\cdots+l_m=k}} \frac{k!}{l_1!\cdots l_m!}[\Gamma^{(U(t))^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(t),\dots,\tilde{U}^{(l_m)}(t)).$$

Remark 5.2. It follows from the boundedness of multiple operator integrals given by Theorem 3.3 that if $\tilde{U}$ has bounded derivatives, then so does φ.

Remark 5.3. (1) Once the formula for the derivatives of φ has been established, it is easy to check, by induction, that the operator Taylor remainder defined by

$$R_{n,f,U}(t) := f(U(t)) - f(U(0)) - \sum_{k=1}^{n-1} \frac{1}{k!}\varphi^{(k)}(0)$$

satisfies, for any $t \in \mathbb{R}$,

$$R_{n,f,U}(t) = \sum_{m=1}^{n} \sum_{\substack{l_1,\dots,l_m\geq 1\\ l_1+\cdots+l_m=n}} [\Gamma^{U(t),(U(0))^m}(f^{[m]})]\left(R_{l_1,U}(t), \frac{\tilde{U}^{(l_2)}(0)}{l_2!}, \dots, \frac{\tilde{U}^{(l_m)}(0)}{l_m!}\right),$$

where $R_{1,U}(t) := \tilde{U}(t)$ and for any $l_1 \geq 2$,

$$R_{l_1,U}(t) := \tilde{U}(t) - \sum_{k=1}^{l_1-1} \frac{1}{k!}\tilde{U}^{(k)}(0).$$

We refer to the proof of [5, Proposition 3.5(ii)] for more details and references.

(2) Theorem 5.1 applies in particular to the function $U(t) = e^{itA}U$ with $U \in \mathcal{U}(\mathcal{H})$ and $A \in \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$, and we retrieve [5, Corollary 3.6] in the more general case of a function f with (not necessarily continuous) bounded nth derivative. In particular, with the same proof as for [5, Corollary 3.6], we obtain

$$\|R_{n,f,U}(1)\|_{p/n} \leq \tilde{c}_{p,n} \sum_{m=1}^{n} \|f^{(m)}\|_\infty \|A\|_p^n, \tag{5.2}$$

where $\tilde{c}_{p,n}$ is a positive constant depending on p and n.

To prove Theorem 5.1, we will carefully approximate, on a subspace of the Hilbert space $\mathcal{H}$, the unitary operator $U(0)$ by another unitary whose spectrum is not the whole $\mathbb{T}$, in order to use Proposition 4.4. The relevant definitions and the first properties of the approximation are given in Section 5.1. The key auxiliary results from Lemma 5.5 will detail the regularity of this approximation process.

5.1. Approximation of unitaries. Let $V \in \mathcal{U}(\mathcal{H})$. For every $j \geq 1$, define $A_j := \{e^{2i\pi t} \mid 0 \leq t \leq (j-1)/j\}$ and set $P_j := E^V(A_j)$. Then $(P_j)_{j\geq 1}$ is an increasing sequence of selfadjoint projections which converges strongly to $I_{\mathcal{H}}$. Recall that this implies that for every $K \in \mathcal{S}^p(\mathcal{H})$, P_jK, KP_j and P_jKP_j converge to K in $\mathcal{S}^p$ as $j \to \infty$. Moreover, P_j commutes with V and the operator $V_j := P_jVP_j = P_jV = VP_j$ is unitary on the Hilbert space $\mathcal{H}_j := P_j\mathcal{H}$ and its spectrum satisfies $\sigma(V_j) \subset A_j$.

Note that if $K \in \mathcal{S}^p(\mathcal{H})$, then $P_jKP_j \in \mathcal{S}^p(\mathcal{H})$ with $\|P_jKP_j\|_p \leq \|K\|_p$ and we can see P_jKP_j as an element of $\mathcal{S}^p(\mathcal{H}_j)$. Similarly, if $X \in \mathcal{S}^p(\mathcal{H}_j)$, we can extend X on $\mathcal{H}$ and keep denoting this operator by X, and in that case $P_jXP_j = X_{|\mathcal{H}_j} \oplus 0_{\mathcal{H}_j^\perp}$.

PROPOSITION 5.4. *Let $1 < p < \infty$. Let $A \in \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$ and define $A_j := P_jAP_j$. Let $n \in \mathbb{N}_{\geq 2}$ and let $f : \mathbb{T} \to \mathbb{C}$ be n-times differentiable such that $f^{(n)}$ is bounded.*

(1) *For every* $K_1, \ldots, K_{n-1} \in \mathcal{S}^p(\mathcal{H})$ *and every* $j \in \mathbb{N}$,

$$[\Gamma^{(e^{iA_j}V_j)^n}(f^{[n-1]})](K_{1,j}, \ldots, K_{n-1,j}) \\ = [\Gamma^{(e^{iA_j}V)^n}(f^{[n-1]})](K_{1,j}, \ldots, K_{n-1,j}),$$

where $K_{i,j} := P_j K_i P_j$.

(2) *For every* $K_1, \ldots, K_n \in \mathcal{S}^p(\mathcal{H})$,

$$[\Gamma^{(VP_j)^{n+1}}(f^{[n]})](K_{1,j}, \ldots, K_{n,j}) \xrightarrow[j\to\infty]{\|\cdot\|_p} [\Gamma^{(V)^{n+1}}(f^{[n]})](K_1, \ldots, K_n).$$

Proof. Let us prove (1). Recall that since $f^{[n-1]}$ is bounded, we have, by construction, $\Gamma^{(e^{iA_j}V_j)^n}(f^{[n-1]}) \in \mathcal{B}_{n-1}(\mathcal{S}^2(\mathcal{H}))$. Let us first etablish the formula when $K_1, \ldots, K_{n-1} \in \mathcal{S}^2(\mathcal{H})$. Since $f^{[n-1]}$ is continuous on $\mathbb{T}^n$ it is sufficient, by a simple approximation argument, to prove the formula when $f^{[n-1]}$ is replaced by a trigonometric polynomial φ on $\mathbb{T}^n$ and by linearity, we can assume that $\varphi = f_1 \otimes \cdots \otimes f_n$, where for any $1 \leq i \leq n, f_i$ is a trigonometric polynomial on $\mathbb{T}$. Since P_j commutes with e^{iA_j} and with V, it is easy to check that for every $1 \leq i \leq n$,

$$f_i(e^{iA_j}V_j)P_j = P_j f_i(e^{iA_j}V_j) = P_j f_i(e^{iA_j}V) = f_i(e^{iA_j}V)P_j.$$

It follows that

$$\begin{aligned}&[\Gamma^{(e^{iA_j}V_j)^n}(\varphi)](K_{1,j}, \ldots, K_{n-1,j}) \\ &\quad= f_1(e^{iA_j}V_j)P_jK_1P_jf_2(e^{iA_j}V_j)P_jK_2P_j \cdots P_jK_{n-1}P_jf_n(e^{iA_m}V_j) \\ &\quad= f_1(e^{iA_j}V)P_jK_1P_jf_2(e^{iA_j}V)P_jK_2P_j \cdots P_jK_{n-1}P_jf_n(e^{iA_j}V) \\ &\quad= [\Gamma^{(e^{iA_j}V)^n}(\varphi)](K_{1,j}, \ldots, K_{n-1,j}).\end{aligned}$$

This proves the formula for $\varphi = f^{[n-1]}$ and $K_1, \ldots, K_{n-1} \in \mathcal{S}^2(\mathcal{H})$. In particular, the formula holds true when $K_i \in \mathcal{S}^2 \cap \mathcal{S}^p$, and approximating (in the $\mathcal{S}^p$-norm) any $K_i \in \mathcal{S}^p$ by a sequence of elements of $\mathcal{S}^2 \cap \mathcal{S}^p$ and using the fact that $\Gamma^{(e^{iA_j}V_j)^n}(f^{[n-1]}), \Gamma^{(e^{iA_j}V)^n}(f^{[n-1]}) \in \mathcal{B}_{n-1}(\mathcal{S}^p(\mathcal{H}))$, we obtain the desired formula.

For the proof of (2), we only make some minor changes: we first etablish that for every $K_1, \ldots, K_n \in \mathcal{S}^2(\mathcal{H})$,

$$[\Gamma^{(VP_j)^{n+1}}(f^{[n]})](K_{1,j}, \ldots, K_{n,j}) = [\Gamma^{(V)^n}(f^{[n]})](K_{1,j}, \ldots, K_{n,j}). \tag{5.3}$$

Since $f^{[n]} \in L^\infty(\prod_{i=1}^{n+1} \lambda_V)$, by the w^*-density of $L^\infty(\lambda_V) \otimes \cdots \otimes L^\infty(\lambda_V)$ in $L^\infty(\prod_{i=1}^{n+1} \lambda_V)$ and the w^*-continuity of multiple operator integrals (see the paragraph before Definition 3.1), it is sufficient to prove the identity when $f^{[n]}$ is replaced by $\varphi \in L^\infty(\lambda_V) \otimes \cdots \otimes L^\infty(\lambda_V)$, and by linearity, we can further assume $\varphi = f_1 \otimes \cdots \otimes f_{n+1}$, where for any $1 \leq i \leq n+1$, $f_i \in L^\infty(\lambda_V)$. Note that $VP_j = V\chi_{A_j}(V) = g_i(V)$ where $g_i(x) = x\chi_{A_j}(x)$ and it is straightforward to check that $(f_i \circ g_i)\chi_{A_j} = f_i \chi_{A_j}$. By [14, Corollary 5.6.29], we have

$$\begin{aligned} f_i(VP_j)P_j &= (f_i \circ g_i)(V)\chi_{A_j}(V) = ((f_i \circ g_i)\chi_{A_j})(V) = f_i(V)\chi_{A_j}(V) \\ &= f_i(V)P_j, \end{aligned}$$

and similarly, $P_j f_i(VP_j) = P_j f_i(V)$. The same computations performed to prove (1) show that (5.3) holds true. Moreover, this formula extends, as before, when $K_1, \dots, K_n \in \mathcal{S}^p(\mathcal{H})$. Finally, the fact that $K_{i,j} \to K_i$ in $\mathcal{S}^p(\mathcal{H})$ for every $1 \le i \le n$ together with $\Gamma^{(V)^{n+1}}(f^{[n]}) \in \mathcal{B}_n(\mathcal{S}^p(\mathcal{H}))$ yield

$$\begin{aligned} [\Gamma^{(VP_j)^{n+1}}(f^{[n]})](K_{1,j}, \dots, K_{n,j}) &= [\Gamma^{(V)^{n+1}}(f^{[n]})](K_{1,j}, \dots, K_{n,j}) \\ &\underset{m\to\infty}{\longrightarrow} [\Gamma^{(V)^{n+1}}(f^{[n]})](K_1, \dots, K_n) \end{aligned}$$

in $\mathcal{S}^p(\mathcal{H})$, which concludes the proof. ∎

5.2. Proof of the main result. In this subsection, we will prove Theorem 5.1. First of all, we need the following lemma. We postpone its proof to the end of the paper to avoid repeating certain arguments and computations which, for some of them, will be very similar to those in the proof of Theorem 5.1.

LEMMA 5.5. *Let $1 < p < \infty$ and $n \in \mathbb{N}$. Let $V \in \mathcal{U}(\mathcal{H})$. Let $\mathcal{A} : \mathbb{R} \to \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$ be n-times differentiable on $\mathbb{R}$ with $\mathcal{A}(0) = 0$. Define*

$$U(t) := e^{i\mathcal{A}(t)}V \quad \text{and} \quad \tilde{U}(t) := e^{i\mathcal{A}(t)}V - V \in \mathcal{S}^p(\mathcal{H}),$$

and, for every $j \in \mathbb{N}$,

$$\mathcal{A}_j(t) := P_j\mathcal{A}(t)P_j,\ U_j(t) := e^{i\mathcal{A}_j(t)}V_j \quad \text{and} \quad \widetilde{U_j}(t) := U_j(t) - U_j(0) \in \mathcal{S}^p(\mathcal{H}),$$

where P_j and $V_j = VP_j$ are defined at the beginning of Section 5.1. *Then we have the following properties:*

(1) *For every $\epsilon > 0$, there exist $J \in \mathbb{N}$ and $\alpha > 0$ such that*

$$\forall j \ge J,\ \forall |t| < \alpha, \quad \|e^{i\mathcal{A}(t)}V - e^{i\mathcal{A}_j(t)}V\|_p \le \epsilon|t|. \tag{5.4}$$

(2) *There exist $\alpha > 0$ and a constant $C > 0$ such that*

$$\forall j \in \mathbb{N},\ \forall |t| < \alpha, \quad \|e^{i\mathcal{A}(t)}V - V\|_p \le C|t| \text{ and } \|e^{i\mathcal{A}_j(t)}V - V\|_p \le C|t|. \tag{5.5}$$

(3) *For every $j \in \mathbb{N}$, $\tilde{U}$ and $\widetilde{U_j}$ are n-times differentiable on $\mathbb{R}$ and for every $0 \le k \le n$ and every $t \in \mathbb{R}$,*

$$P_j\widetilde{U_j}^{(k)}(t)P_j = \widetilde{U_j}^{(k)}(t). \tag{5.6}$$

(4) *For every $\epsilon > 0$, there exist $J \in \mathbb{N}$ and $\alpha > 0$ such that, for every $1 \le k \le n-1$,*

$$\forall j \ge J,\ \forall |t| < \alpha, \quad \|\tilde{U}^{(k)}(t) - \widetilde{U_j}^{(k)}(t)\|_p \le \epsilon. \tag{5.7}$$

(5) *Let* $0 \le k \le n-1$. *Then we can write*

$$\tilde{U}^{(k)}(t) = \tilde{U}^{(k)}(0) + tR_k(t) \quad \text{and} \quad \widetilde{U_j}^{(k)}(t) = \widetilde{U_j}^{(k)}(0) + tR_k^j(t),$$

where, for every $\epsilon > 0$, *there exist* $J \in \mathbb{N}$ *and* $\alpha > 0$ *such that*

$$\forall j \ge J, \forall |t| < \alpha, \qquad \|R_k(t) - R_k^j(t)\|_p \le \epsilon. \tag{5.8}$$

Proof of Theorem 5.1. The proof will be divided into three steps. First, we show that we can rewrite the function U in a convenient way. Next, we will approximate the unitary $U(0)$ in order to use Proposition 4.4. Finally, thanks to several estimates that will be using Lemma 5.5, we will obtain the result.

STEP 1. *Simplification of the function* U. First of all, note that by translation, it is sufficient to prove the result for $t = 0$. Let

$$V := U(0) \in \mathcal{U}(\mathcal{H}).$$

By continuity of U, $U(t)V^* \to I_{\mathcal{H}}$ as $t \to 0$ in the operator norm, so that, for $t \in I$ where I is a real interval centered at $t = 0$, we have $\|U(t)V^* - I_{\mathcal{H}}\| < 1/2$. In particular, we can set $\mathcal{A}(t) := -i\log(U(t)V^*)$ and we find that $\mathcal{A}(t) \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$. This function satisfies $\mathcal{A}(0) = 0$ and $e^{i\mathcal{A}(t)}V = U(t)$. Moreover, the assumption $U(t) - U(0) \in \mathcal{S}^p(\mathcal{H})$ implies that $\mathcal{A}(t) \in \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$, and since $\mathcal{A}(t)$ is obtained by means of a power series, the fact that $\tilde{U} : \mathbb{R} \to \mathcal{S}^p(\mathcal{H})$ is n-times differentiable on $\mathbb{R}$ implies that $\mathcal{A} : I \to \mathcal{S}^p_{\mathrm{sa}}(\mathcal{H})$ is n-times differentiable on I. Alternatively, since log is C^∞ in a neighborhood of 1, this follows from [5, Theorem 3.5]. Hence, from now on, we will assume that

$$\forall t \in I, \qquad U(t) = e^{i\mathcal{A}(t)}V,$$

where $\mathcal{A}$ has the properties given above.

STEP 2. *Initiation of the approximation process.* For every $j \ge 1$, define $A_j := \{e^{2i\pi t} \mid 0 \le t \le (j-1)/j\}$ and set, as in Section 5.1,

$$P_j := E^V(A_j).$$

Recall that the operator $V_j := P_jVP_j = P_jV = VP_j$ is unitary on the Hilbert space $\mathcal{H}_j := P_j\mathcal{H}$ and its spectrum satisfies $\sigma(V_j) \subset A_j$. Define

$$\mathcal{A}_j(t) := P_j\mathcal{A}(t)P_j, \quad U_j(t) := e^{i\mathcal{A}_j(t)}V_j \quad \text{and} \quad \widetilde{U_j}(t) := U_j(t) - U_j(0) \in \mathcal{S}^p,$$

where $\mathcal{A}_j(t)$ and $\widetilde{U_j}(t)$ can be seen as elements of either $\mathcal{S}^p(\mathcal{H}_j)$ or $\mathcal{S}^p(\mathcal{H})$.

Now, define

$$\varphi_j : \mathbb{R} \ni t \mapsto f(U_j(t)) - f(U_j(0)) \in \mathcal{S}^p(\mathcal{H}_j).$$

The operator $\varphi_j(t)$ acts as well on $\mathcal{H}$ and is equal to 0 on $\mathcal{H}_j^\perp$. Since $e^{i\mathcal{A}_j(t)}V_j \in \mathcal{U}(\mathcal{H}_j)$ and $\sigma(V_j) \subset A_j$ and hence $\sigma(V_j) \neq \mathbb{T}$, by Proposition 4.4

and Remark 4.5, φ_j is n-times differentiable in a neighborhood of 0, which we can assume to be equal to I, so that we can write, for every $t \in I$,

$$\varphi_j^{(n-1)}(t) - \varphi_j^{(n-1)}(0) - t\varphi_j^{(n)}(0) = o_j(t), \tag{5.9}$$

where $o_j(t) = o(t)$ depends on j, and where $\varphi_j^{(n-1)}$ and $\varphi_j^{(n)}$ are given by formula (5.1).

Since $f \in C^{n-1}(\mathbb{T})$, by [5, Theorem 3.5], φ is $(n-1)$-times differentiable on $\mathbb{R}$ and for every $t \in \mathbb{R}$,

$$\varphi^{(n-1)}(t) = \sum_{m=1}^{n-1} \sum_{\substack{l_1,\dots,l_m \geq 1 \\ l_1+\dots+l_m=n-1}} \frac{(n-1)!}{l_1!\cdots l_m!}[\Gamma^{(U(t))^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(t),\dots,\tilde{U}^{(l_m)}(t)).$$

Let us define

$$T := \sum_{m=1}^{n} \sum_{\substack{l_1,\dots,l_m \geq 1 \\ l_1+\dots+l_m=n}} \frac{n!}{l_1!\cdots l_m!}[\Gamma^{(V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(0),\dots,\tilde{U}^{(l_m)}(0)),$$

$$T_j := \sum_{m=1}^{n} \sum_{\substack{l_1,\dots,l_m \geq 1 \\ l_1+\dots+l_m=n}} \frac{n!}{l_1!\cdots l_m!}[\Gamma^{(V_j)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(0),\dots,\widetilde{U_j}^{(l_m)}(0)).$$

In particular, $\varphi_j^{(n)}(0) = T_j$. To prove the theorem, we have to show that

$$\varphi^{(n-1)}(t) - \varphi^{(n-1)}(0) - tT = o(t) \tag{5.10}$$

as $t \to 0$. Note that if $n = 1$, we do not use [5, Theorem 3.5], and (5.10) reduces to

$$f(U(t)) - f(V) - t[\Gamma^{V,V}(f^{[1]})](\tilde{U}'(0)) = o(t).$$

To prove our claim, let us write, for every $j \in \mathbb{N}$,

$$\begin{aligned} &\varphi^{(n-1)}(t) - \varphi^{(n-1)}(0) - tT \\ &\qquad = L_j(t) - t(T - T_j) + \big(\varphi_j^{(n-1)}(t) - \varphi_j^{(n-1)}(0) - t\varphi_j^{(n)}(0)\big), \end{aligned} \tag{5.11}$$

where

$$L_j(t) := \varphi^{(n-1)}(t) - \varphi_j^{(n-1)}(t) + \varphi_j^{(n-1)}(0) - \varphi^{(n-1)}(0). \tag{5.12}$$

First, we will estimate the quantity $T - T_j$ uniformly for j large enough, and secondly, we will estimate the term $L_j(t)$ for t small enough and j large enough. Eventually, we will use (5.9) to estimate the last term appearing in (5.11), for a fixed integer j.

Step 3. *Estimates in the approximation process.* Let us fix $\epsilon > 0$.

Estimate of $T - T_j$. Let $1 \leq m \leq n$ and let $l_1, \ldots, l_m \geq 1$ be such that $l_1 + \cdots + l_m = n$. According to Proposition 5.4(2), if $j \geq J^1$ is large enough,

$$\begin{aligned}\|[\Gamma^{(V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(0), \ldots, \tilde{U}^{(l_m)}(0)) \\ - [\Gamma^{(V_j)^{m+1}}(f^{[m]})](P_j\tilde{U}^{(l_1)}(0)P_j, \ldots, P_j\tilde{U}^{(l_m)}(0)P_j)\|_p \leq \epsilon,\end{aligned}$$

and, according to Lemma 5.5(3)(4), and to the uniform boundedness of $(\Gamma^{(V_j)^{m+1}}(f^{[m]}))_j$, if $j \geq J^2$ is large enough,

$$\begin{aligned}\|[\Gamma^{(V_j)^{m+1}}(f^{[m]})](P_j\tilde{U}^{(l_1)}(0)P_j, \ldots, P_j\tilde{U}^{(l_m)}(0)P_j) \\ - [\Gamma^{(V_j)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(0), \ldots, \widetilde{U_j}^{(l_m)}(0))\|_p \leq \epsilon.\end{aligned}$$

It follows that for every $j \geq \max\{J^1, J^2\} =: J$,

$$\begin{aligned}\|[\Gamma^{(V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(0), \ldots, \tilde{U}^{(l_m)}(0)) \\ - [\Gamma^{(V_j)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(0), \ldots, \widetilde{U_j}^{(l_m)}(0))\|_p \leq 2\epsilon.\end{aligned}$$

Hence, since T and T_j are finite sums of such terms, there exist a constant c_0 and an integer J_0 such that

$$\forall j \geq J_0, \quad \|T - T_j\|_p \leq c_0\epsilon. \tag{5.13}$$

Estimate of $L_j(t)$. Recall that

$$\begin{aligned}&\varphi_j^{(n-1)}(t) \\ &= \sum_{m=1}^{n-1} \sum_{\substack{l_1,\ldots,l_m\geq 1\\ l_1+\cdots+l_m=n-1}} \frac{(n-1)!}{l_1!\cdots l_m!}[\Gamma^{(U_j(t))^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(t), \ldots, \widetilde{U_j}^{(l_m)}(t)) \\ &= \sum_{m=1}^{n-1} \sum_{\substack{l_1,\ldots,l_m\geq 1\\ l_1+\cdots+l_m=n-1}} \frac{(n-1)!}{l_1!\cdots l_m!}[\Gamma^{(e^{i\mathcal{A}_j(t)}V)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(t), \ldots, \widetilde{U_j}^{(l_m)}(t)),\end{aligned}$$

where the last equality follows from Lemma 5.5(3) and Proposition 5.4(1). When $t = 0$, $e^{i\mathcal{A}_j(0)}V = V$ so we have

$$\begin{aligned}&\varphi_j^{(n-1)}(0) \\ &\quad= \sum_{m=1}^{n-1} \sum_{\substack{l_1,\ldots,l_m\geq 1\\ l_1+\cdots+l_m=n-1}} \frac{(n-1)!}{l_1!\cdots l_m!}[\Gamma^{(V)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(0), \ldots, \widetilde{U_j}^{(l_m)}(0)).\end{aligned}$$

Fix $1 \leq m \leq n-1$ and $l_1, \ldots, l_m \geq 1$ such that $l_1 + \cdots + l_m = n-1$. From the expression (5.12) of $L_j(t)$, according to the latter and by linearity, if we show that

$$\begin{aligned}Q_j(t) := {} & [\Gamma^{(e^{i\mathcal{A}(t)}V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(t),\dots,\tilde{U}^{(l_m)}(t)) \\ & - [\Gamma^{(e^{i\mathcal{A}_j(t)}V)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(t),\dots,\widetilde{U_j}^{(l_m)}(t)) \\ & + [\Gamma^{(V)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(0),\dots,\widetilde{U_j}^{(l_m)}(0)) \\ & - [\Gamma^{(V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(0),\dots,\tilde{U}^{(l_m)}(0))\end{aligned}$$

satisfies $\|Q_j(t)\|_p \le c\epsilon|t|$ for some constant c depending only on p, m, $l_1,\dots,l_m$, f and where j is large enough and t small enough, a similar inequality will hold true for $\|L_j(t)\|_p$.

Let us write

$$Q_j(t) = S_{1,j}(t) + S_{2,j}(t) + S_{3,j}(t),$$

where

$$\begin{aligned}S_{1,j}(t) &= [\Gamma^{(e^{i\mathcal{A}(t)}V)^{m+1}}(f^{[m]}) - \Gamma^{(e^{i\mathcal{A}_j(t)}V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(t),\dots,\tilde{U}^{(l_m)}(t)), \\ S_{2,j}(t) &= [\Gamma^{(e^{i\mathcal{A}_j(t)}V)^{m+1}}(f^{[m]}) - \Gamma^{(V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(t),\dots,\tilde{U}^{(l_m)}(t)) \\ &\quad + [\Gamma^{(V)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(t),\dots,\widetilde{U_j}^{(l_m)}(t)) \\ &\quad - [\Gamma^{(e^{i\mathcal{A}_j(t)}V)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(t),\dots,\widetilde{U_j}^{(l_m)}(t)), \\ S_{3,j}(t) &= [\Gamma^{(V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(t),\dots,\tilde{U}^{(l_m)}(t)) \\ &\quad - [\Gamma^{(V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(0),\dots,\tilde{U}^{(l_m)}(0)) \\ &\quad + [\Gamma^{(V)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(0),\dots,\widetilde{U_j}^{(l_m)}(0)) \\ &\quad - [\Gamma^{(V)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(t),\dots,\widetilde{U_j}^{(l_m)}(t)).\end{aligned}$$

First, since $\tilde{U}$ is n-times differentiable and $1 \le l_k \le n-1$, the derivatives $\tilde{U}^{(l_k)}$ are $\mathcal{S}^p$-bounded in a neighborhood of 0. Hence, according to Corollary 3.7 and Lemma 5.5(1), there exist a constant c_1, an integer J_1 and $\alpha > 0$ such that

$$\text{(5.14)} \quad \forall j \ge J_1,\ \forall |t| < \alpha, \qquad \|S_{1,j}(t)\|_p \le c_1\|e^{i\mathcal{A}_j(t)}V - e^{i\mathcal{A}(t)}V\| \le c_1\epsilon|t|.$$

Next, according to Corollary 3.6, we have

$$\begin{aligned}S_{2,j}(t) = t\sum_{q=1}^{m+1}\Bigg(& [\Gamma^{(e^{i\mathcal{A}_j(t)}V)^q,(V)^{m-q+1}}(f^{[m+1]})] \\ & \left(\tilde{U}^{(l_1)}(t),\dots,\tilde{U}^{(l_{q-1})}(t),\frac{e^{i\mathcal{A}_j(t)}V - V}{t},\tilde{U}^{(l_q)}(t),\dots,\tilde{U}^{(l_m)}(t)\right) \\ & - [\Gamma^{(e^{i\mathcal{A}_j(t)}V)^q,(V)^{m-q+1}}(f^{[m+1]})] \\ & \left(\widetilde{U_j}^{(l_1)}(t),\dots,\widetilde{U_j}^{(l_{q-1})}(t),\frac{e^{i\mathcal{A}_j(t)}V - V}{t},\widetilde{U_j}^{(l_q)}(t),\dots,\widetilde{U_j}^{(l_m)}(t)\right)\Bigg).\end{aligned}$$

According to Lemma 5.5(2)(4), there exist $\beta > 0$ and an integer $J_2 \in \mathbb{N}$ such that, for every $1 \le k \le m$, every $j \ge J_2$ and every $|t| < \beta$,

$$\|\tilde{U}^{(l_k)}(t) - \widetilde{U_j}^{(l_k)}(t)\|_p \le \epsilon \quad \text{and} \quad \left\| \frac{e^{i\mathcal{A}_j(t)}V - V}{t} \right\|_p \text{ is bounded.}$$

Since $\tilde{U}^{(l_k)}(t)$ and $\widetilde{U_j}^{(l_k)}(t)$, $1 \le k \le m$, are locally bounded around 0, and since the operators $\Gamma^{(e^{i\mathcal{A}_j(t)}V)^q,(V)^{m-q+1}}(f^{[m+1]})$ are uniformly bounded with respect to t, there exists a constant C depending on p, m, f and U such that

$$\forall j \ge J_2,\ \forall |t| < \beta, \qquad \|S_{2,j}(t)\|_p \le |t| \sum_{i=1}^{m+1} C\epsilon =: c_2 \epsilon |t|. \tag{5.15}$$

Finally, to estimate $S_{3,j}(t)$, let us write, according to Lemma 5.5(5),

$$\tilde{U}^{(l_k)}(t) = \tilde{U}^{(l_k)}(0) + tR_{l_k}(t) \quad \text{and} \quad \widetilde{U_j}^{(l_k)}(t) = \widetilde{U_j}^{(l_k)}(0) + tR^j_{l_k}(t).$$

It follows that

$$\begin{aligned} &[\Gamma^{(V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(t), \ldots, \tilde{U}^{(l_m)}(t)) \\ &\qquad\qquad - [\Gamma^{(V)^{m+1}}(f^{[m]})](\tilde{U}^{(l_1)}(0), \ldots, \tilde{U}^{(l_m)}(0)) \\ &\qquad = \sum_{\substack{A_i(t) \in \{\tilde{U}^{(l_i)}(0),\, tR_{l_i}(t)\} \\ \exists 1 \le i \le m,\, A_i(t) = tR_{l_i}(t)}} [\Gamma^{(V)^{m+1}}(f^{[m]})](A_1(t), \ldots, A_m(t)), \end{aligned}$$

and similarly

$$\begin{aligned} &[\Gamma^{(V)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(0), \ldots, \widetilde{U_j}^{(l_m)}(t)) \\ &\qquad\qquad - [\Gamma^{(V)^{m+1}}(f^{[m]})](\widetilde{U_j}^{(l_1)}(t), \ldots, \widetilde{U_j}^{(l_m)}(t)) \\ &\qquad = \sum_{\substack{B^j_i(t) \in \{\widetilde{U_j}^{(l_i)}(0),\, tR^j_{l_i}(t)\} \\ \exists 1 \le i \le m,\, B^j_i(t) = tR^j_{l_i}(t)}} [\Gamma^{(V)^{m+1}}(f^{[m]})](B^j_1(t), \ldots, B^j_m(t)). \end{aligned}$$

Hence, we only have to estimate the terms

$$[\Gamma^{(V)^{m+1}}(f^{[m]})](A_1(t), \ldots, A_m(t)) - [\Gamma^{(V)^{m+1}}(f^{[m]})](B^j_1(t), \ldots, B^j_m(t)),$$

where $A_i(t) = \tilde{U}^{(l_i)}(0)$ if and only if $B^j_i(t) = \widetilde{U_j}^{(l_i)}(0)$. Moreover, to simplify the notations, we assume that $A_1(t) = tR_{l_1}(t)$ and $B^j_1(t) = tR^j_{l_1}(t)$. In that case,

$$
\begin{aligned}
&[\Gamma^{(V)^{m+1}}(f^{[m]})](A_1(t),\dots,A_m(t)) - [\Gamma^{(V)^{m+1}}(f^{[m]})](B_1(t),\dots,B_m(t)) \\
&\qquad = t\big([\Gamma^{(V)^{m+1}}(f^{[m]})](R_{l_1}(t),A_2(t),\dots,A_m(t)) \\
&\qquad\qquad - [\Gamma^{(V)^{m+1}}(f^{[m]})](R^j_{l_1}(t),B^j_2(t),\dots,B^j_m(t))\big).
\end{aligned}
$$

According to Lemma 5.5(4)(5), there exist an integer J_3' and $\gamma' > 0$ such that, for every $2 \le i \le m$,

$$\forall j \ge J_3', \forall |t| < \gamma', \quad \|R_{l_1}(t) - R^j_{l_1}(t)\| \le \epsilon \quad \text{and} \quad \|A_i(t) - B^j_i(t)\|_p \le \epsilon.$$

It follows that there exists a constant $C' > 0$ such that

$$
\begin{aligned}
&\|[\Gamma^{(V)^{m+1}}(f^{[m]})](A_1(t),\dots,A_m(t)) - [\Gamma^{(V)^{m+1}}(f^{[m]})](B^j_1(t),\dots,B^j_m(t))\|_p \\
&\qquad \le C'\epsilon|t|.
\end{aligned}
$$

In particular, there exist an integer J_3, $\gamma > 0$ and a constant $c_3 > 0$ such that

$$\forall j \ge J_3, \forall |t| < \gamma, \quad \|S_{3,j}(t)\|_p \le c_3\epsilon|t|. \tag{5.16}$$

Setting $c = c_1 + c_2 + c_3$, $\delta = \min\{\alpha,\beta,\gamma\}$ and $J := \max\{J_1,J_2,J_3\}$, we deduce from (5.14)–(5.16) that

$$\forall j \ge J, \forall |t| < \delta, \quad \|Q_j(t)\|_p \le c\epsilon|t|. \tag{5.17}$$

From the definition (5.12) of $L_j(t)$, it follows that there exist $J' \in \mathbb{N}$, $\delta' > 0$ and a constant $c' > 0$ such that

$$\forall j \ge J', \forall |t| < \delta', \quad \|L_j(t)\|_p \le c'\epsilon|t|. \tag{5.18}$$

Conclusion. Fix an integer $j_0 \ge \max\{J_0, J'\}$. According to (5.9), there exists $\delta'' > 0$ such that

$$\forall |t| < \delta'', \quad \|\varphi_{j_0}^{(n-1)}(t) - \varphi_{j_0}^{(n-1)}(0) - t\varphi_{j_0}^{(n)}(0)\|_p \le \epsilon|t|.$$

According to (5.13) and (5.18), we deduce from the equality (5.11) that, for every $t \in I$ such that $|t| < \min\{\delta',\delta''\}$,

$$\|\varphi^{(n-1)}(t) - \varphi^{(n-1)}(0) - tT\|_p \le (c' + c_0 + 1)\epsilon|t|.$$

Hence, we proved that

$$\varphi^{(n-1)}(t) - \varphi^{(n-1)}(0) - tT = o(t),$$

which shows that $\varphi^{(n-1)}$ is differentiable at $t = 0$ with $\varphi^{(n)}(0) = T$, and finishes the proof. ∎

We conclude this paper by proving Lemma 5.5.

Proof of Lemma 5.5. To prove (1), note that by Duhamel's formula (see, e.g., [3, Lemma 5.2]), we have

$$\|e^{i\mathcal{A}(t)}V - e^{i\mathcal{A}_j(t)}V\|_p = \|e^{i\mathcal{A}(t)} - e^{i\mathcal{A}_j(t)}\|_p \le \|\mathcal{A}(t) - \mathcal{A}_j(t)\|_p.$$

Recall that $\mathcal{A}(0)=0$, so we can write $\mathcal{A}(t)=t\mathcal{A}'(0)+o(t)$ as $t\to 0$, and hence $\mathcal{A}_j(t)=tP_j\mathcal{A}'(0)P_j+P_jo(t)P_j$. It follows that

$$\|\mathcal{A}(t)-\mathcal{A}_j(t)\|_p \le |t|\,\|\mathcal{A}'(0)-P_j\mathcal{A}'(0)P_j\|_p+\|o(t)-P_jo(t)P_j\|_p.$$

Since $\mathcal{A}'(0)\in\mathcal{S}^p(\mathcal{H})$, for j large enough, we have

$$\|\mathcal{A}'(0)-P_j\mathcal{A}'(0)P_j\|_p\le\epsilon,$$

and for $|t|$ small enough, we have

$$\|o(t)-P_jo(t)P_j\|_p\le 2\|o(t)\|_p\le\epsilon|t|,$$

which gives the desired inequality.

The proof of (2) is similar. Indeed, it suffices to write

$$\|e^{i\mathcal{A}(t)}V-V\|_p=\|e^{i\mathcal{A}(t)}-e^0\|_p\le\|\mathcal{A}(t)\|_p=|t|\left\|\frac{\mathcal{A}(t)}{t}\right\|_p\le C|t|, \tag{5.19}$$

where $C:=2\|\mathcal{A}'(0)\|_p$, for t small enough. The proof of the second inequality in (5.5) is identical.

For the rest of the proof, we let $g:t\mapsto e^{it}$. Then we can write

$$\tilde{U}(t)=[g(\mathcal{A}(t))-g(\mathcal{A}(0))]V \quad\text{and}\quad \widetilde{U_j}(t)=[g(\mathcal{A}_j(t))-g(\mathcal{A}_j(0))]VP_j.$$

Since $g\in C^\infty(\mathbb{R})$ with bounded derivatives, by Corollary 4.2, $\tilde{U}$ and $\widetilde{U_j}$ are n-times differentiable on $\mathbb{R}$ and for every $1\le k\le n$ and every $t\in\mathbb{R}$,

$$\tilde{U}^{(k)}(t)=\Bigg(\sum_{m=1}^{k}\sum_{\substack{l_1,\dots,l_m\ge 1\\ l_1+\dots+l_m=k}}\frac{k!}{l_1!\cdots l_m!}D_{l_1,\dots,l_m}(t)\Bigg)V, \tag{5.20}$$

$$\widetilde{U_j}^{(k)}(t)=\Bigg(\sum_{m=1}^{k}\sum_{\substack{l_1,\dots,l_m\ge 1\\ l_1+\dots+l_m=k}}\frac{k!}{l_1!\cdots l_m!}D^j_{l_1,\dots,l_m}(t)\Bigg)VP_j, \tag{5.21}$$

where

$$\begin{aligned}
D_{l_1,\dots,l_m}(t)&=[\mathit{\Gamma}^{(\mathcal{A}(t))^{m+1}}(g^{[m]})](\mathcal{A}^{(l_1)}(t),\dots,\mathcal{A}^{(l_m)}(t)),\\
D^j_{l_1,\dots,l_m}(t)&=[\mathit{\Gamma}^{(\mathcal{A}_j(t))^{m+1}}(g^{[m]})]((\mathcal{A}_j)^{(l_1)}(t),\dots,(\mathcal{A}_j)^{(l_m)}(t))\\
&=[\mathit{\Gamma}^{(\mathcal{A}_j(t))^{m+1}}(g^{[m]})](P_j\mathcal{A}^{(l_1)}(t)P_j,\dots,P_j\mathcal{A}^{(l_m)}(t)P_j).
\end{aligned}$$

To complete the proof of (3), note that for every $t\in\mathbb{R}$,

$$P_j\widetilde{U_j}(t)P_j=P_j(e^{i\mathcal{A}_j(t)}VP_j-VP_j)P_j=e^{i\mathcal{A}_j(t)}VP_j-VP_j=\widetilde{U_j}(t),$$

which follows from the fact that P_j commute with $e^{i\mathcal{A}_j(t)}$ and V. Hence, differentiating this formula k times gives the result.

Next, according to the latter, to prove (4), we only have to estimate

$$\|D_{l_1,\dots,l_m}(t)V-D^j_{l_1,\dots,l_m}(t)VP_j\|_p$$

for all $1 \leq m \leq n-1$ and $l_1, \ldots, l_m \geq 1$ such that $l_1 + \cdots + l_m \leq n-1$. But it is easy to check that

$$D^j_{l_1,\ldots,l_m}(t)VP_j = D^j_{l_1,\ldots,l_m}(t)P_jV = D^j_{l_1,\ldots,l_m}(t)V,$$

so that

$$\|D_{l_1,\ldots,l_m}(t)V - D^j_{l_1,\ldots,l_m}(t)VP_j\|_p = \|D_{l_1,\ldots,l_m}(t) - D^j_{l_1,\ldots,l_m}(t)\|_p. \tag{5.22}$$

Denote $T_t := [\Gamma^{(\mathcal{A}(t))^{m+1}}(g^{[m]})]$ and $T_{t,j} := [\Gamma^{(\mathcal{A}_j(t))^{m+1}}(g^{[m]})]$. We have

$$\begin{aligned}
&D_{l_1,\ldots,l_m}(t) - D^j_{l_1,\ldots,l_m}(t)\\
&= \big(T_t(\mathcal{A}^{(l_1)}(t), \ldots, \mathcal{A}^{(l_m)}(t)) - T_t(\mathcal{A}^{(l_1)}(0), \ldots, \mathcal{A}^{(l_m)}(0))\big)\\
&\quad + \big(T_t(\mathcal{A}^{(l_1)}(0), \ldots, \mathcal{A}^{(l_m)}(0)) - T_t(P_j\mathcal{A}^{(l_1)}(0)P_j, \ldots, P_j\mathcal{A}^{(l_m)}(0)P_j)\big)\\
&\quad + \big(T_t(P_j\mathcal{A}^{(l_1)}(0)P_j, \ldots, P_j\mathcal{A}^{(l_m)}(0)P_j) - T_{t,j}(P_j\mathcal{A}^{(l_1)}(0)P_j, \ldots, P_j\mathcal{A}^{(l_m)}(0)P_j)\big)\\
&\quad + \big(T_{t,j}(P_j\mathcal{A}^{(l_1)}(0)P_j, \ldots, P_j\mathcal{A}^{(l_m)}(0)P_j) - T_{t,j}(P_j\mathcal{A}^{(l_1)}(t)P_j, \ldots, P_j\mathcal{A}^{(l_m)}(t)P_j)\big)\\
&:= K_1(t) + K_{2,j}(t) + K_{3,j}(t) + K_{4,j}(t).
\end{aligned}$$

The continuity of $\mathcal{A}^{(l_k)}$, $1 \leq k \leq m$, and the uniform boundedness of $(T_t)_{t\in\mathbb{R}}$ give the existence of C_1 (depending on f, $\mathcal{A}$ and p) and $\alpha_1 > 0$ such that

$$\forall |t| < \alpha_1, \quad \|K_1(t)\|_p \leq C_1 \max_{1\leq k\leq m} \|\mathcal{A}^{(l_k)}(t) - \mathcal{A}^{(l_k)}(0)\|_p \leq \epsilon.$$

To estimate $K_{2,j}(t)$, it is enough to notice that since $\mathcal{A}^{(l_k)}(0) \in \mathcal{S}^p(\mathcal{H})$, $1 \leq k \leq m$, we have

$$P_j\mathcal{A}^{(l_k)}(0)P_j \to \mathcal{A}^{(l_k)}(0),$$

in $\mathcal{S}^p$ as $j \to \infty$, so that

$$\|K_{2,j}(t)\| \leq C_1 \max_{1\leq k\leq m} \|\mathcal{A}^{(l_k)}(0) - P_j\mathcal{A}^{(l_k)}(0)P_j\|_p \leq \epsilon$$

for $j \geq J$ large enough. For the third term, by Corollary 3.7 there exists a constant C_2 (depending on f and p) such that

$$\|K_{3,j}(t)\|_p \leq C_2\|\mathcal{A}(t) - \mathcal{A}_j(t)\|_p\|P_j\mathcal{A}^{(l_1)}(0)P_j\| \cdots \|P_j\mathcal{A}^{(l_m)}(0)P_j\|_p \leq \epsilon$$

for j large enough and $|t| < \alpha_2$ small enough, according to the proof of (1). Since $K_{4,j}(t)$ can be estimated like $K_1(t)$, this concludes the proof of (4).

Finally, to prove (5), write, for $0 \leq k \leq n-1$ and $t \neq 0$,

$$R_k(t) := \frac{\tilde{U}^{(k)}(t) - \tilde{U}^{(k)}(0)}{t} \quad \text{and} \quad R^j_k(t) := \frac{\widetilde{U_j}^{(k)}(t) - \widetilde{U_j}^{(k)}(0)}{t},$$

so that

$$\tilde{U}^{(k)}(t) = \tilde{U}^{(k)}(0) + tR_k(t) \quad \text{and} \quad \widetilde{U_j}^{(k)}(t) = \widetilde{U_j}^{(k)}(0) + tR^j_k(t).$$

Using the same notations as before, it follows from (5.20)–(5.22) that to estimate

$$\|R_k(t) - R_k^j(t)\|_p,$$

it suffices to estimate the quantity

$$\frac{1}{|t|}\|(D_{l_1,\ldots,l_m}(t) - D_{l_1,\ldots,l_m}(0)) - (D^j_{l_1,\ldots,l_m}(t) - D^j_{l_1,\ldots,l_m}(0))\|_p$$

for all $1 \leq m \leq n-1$ and $l_1, \ldots, l_m \geq 1$ such that $l_1 + \cdots + l_m \leq n-1$. To do so, let us write, with the notations T_t and $T_{t,j}$ introduced above,

$$\begin{aligned}
&(D_{l_1,\ldots,l_m}(t) - D_{l_1,\ldots,l_m}(0)) - (D^j_{l_1,\ldots,l_m}(t) - D^j_{l_1,\ldots,l_m}(0)) \\
&= \big((T_t - T_{t,j})(P_j\mathcal{A}^{(l_1)}(t)P_j, \ldots, P_j\mathcal{A}^{(l_m)}(t)P_j)\big) \\
&\quad + \big((T_t - T_0)(\mathcal{A}^{(l_1)}(t), \ldots, \mathcal{A}^{(l_m)}(t)) - (T_t - T_0)(P_j\mathcal{A}^{(l_1)}(t)P_j, \ldots, P_j\mathcal{A}^{(l_m)}(t)P_j)\big) \\
&\quad + [T_0(P_j\mathcal{A}^{(l_1)}(0)P_j, \ldots, P_j\mathcal{A}^{(l_m)}(0)P_j) - T_0(P_j\mathcal{A}^{(l_1)}(t)P_j, \ldots, P_j\mathcal{A}^{(l_m)}(t)P_j) \\
&\quad + T_0(\mathcal{A}^{(l_1)}(t), \ldots, \mathcal{A}^{(l_m)}(t)) - T_0(\mathcal{A}^{(l_1)}(0), \ldots, \mathcal{A}^{(l_m)}(0))].
\end{aligned}$$

Denote by $L_1^j(t)$ and $L_2^j(t)$ the quantities on the first two lines in the last equality, and by $L_3^j(t)$ the quantity on the last two lines.

For L_1^j, by the boundedness of $\mathcal{A}^{(l_k)}(t)$, $1 \leq k \leq m$, in a neighborhood of 0 and by Corollary 3.7, there exists $D_1 > 0$ such that

$$\|L_1^j(t)\|_p \leq D_1\|\mathcal{A}_j(t) - \mathcal{A}(t)\|_p \leq \epsilon|t|$$

for j large enough and $|t|$ small enough, according to item (1).

For the term L_2^j, by Corollary 3.7 and (5.19), there exists of $D_2 > 0$ such that

$$\|T_t - T_0\|_{\mathcal{B}_m(\mathcal{S}^p(\mathcal{H}))} \leq D_2\|\mathcal{A}(t)\|_p \leq D_2'|t|$$

for t small enough and for some constant D_2'. Using again the boundedness of $\mathcal{A}^{(l_k)}(t)$, $1 \leq k \leq m$, in a neighborhood of 0, we get the existence of D_2'' such that

$$\|L_2^j(t)\|_p \leq D_2''|t| \max_{1\leq k\leq m} \|\mathcal{A}^{(l_k)}(t) - P_j\mathcal{A}^{(l_k)}(t)P_j\|_p \leq \epsilon|t|,$$

where the last inequality is obtained by writing

$$\mathcal{A}^{(l_k)}(t) = \mathcal{A}^{(l_k)}(0) + t\mathcal{A}^{(l_k+1)}(0) + o(t),$$

and applying the same computations as in item (1).

Finally, let us write, for each $1 \leq k \leq m$,

$$\mathcal{A}^{(l_k)}(t) = \mathcal{A}^{(l_k)}(0) + t\frac{\mathcal{A}^{(l_k)}(t) - \mathcal{A}^{(l_k)}(0)}{t} =: \mathcal{A}^{(l_k)}(0) + tG_k(t),$$

and

$$P_j\mathcal{A}^{(l_k)}(t)P_j = P_j\mathcal{A}^{(l_k)}(0)P_j + tP_jG_k(t)P_j.$$

We have

$$\begin{aligned}
&\|G_k(t) - P_jG_k(t)P_j\|_p \\
&\quad \leq \left\| \frac{\mathcal{A}^{(l_k)}(t) - \mathcal{A}^{(l_k)}(0)}{t} - \mathcal{A}^{(l_k+1)}(0) \right\|_p + \|\mathcal{A}^{(l_k+1)}(0) - P_j\mathcal{A}^{(l_k+1)}(0)P_j\|_p \\
&\qquad + \left\| P_j \frac{\mathcal{A}^{(l_k)}(t) - \mathcal{A}^{(l_k)}(0)}{t} P_j - P_j\mathcal{A}^{(l_k+1)}(0)P_j \right\|_p \\
&\quad \leq 2\left\| \frac{\mathcal{A}^{(l_k)}(t) - \mathcal{A}^{(l_k)}(0)}{t} - \mathcal{A}^{(l_k+1)}(0) \right\|_p + \|\mathcal{A}^{(l_k+1)}(0) - P_j\mathcal{A}^{(l_k+1)}(0)P_j\|_p \\
&\quad \leq \epsilon
\end{aligned}$$

for j large enough and t small enough. Now, following the same computations as those used to estimate the term $S_{3,j}(t)$ in the proof of Theorem 5.1, we obtain, taking larger j and smaller $|t|$ if necessary, the estimate

$$\|L_3^j(t)\| \leq \epsilon|t|.$$

In particular, we have proved that there exist $J \in \mathbb{N}$ and $\alpha > 0$ such that

$$\forall j \geq J, \forall |t| < \alpha, \quad \|R_k(t) - R_k^j(t)\|_p \leq \epsilon.$$

This concludes the proof of the lemma. ∎

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References

[1] J. Arazy, T. J. Barton and Y. Friedman, *Operator differentiable functions*, Integral Equations Operator Theory 13 (1990), 462–487.

[2] P. J. Ayre, M. G. Cowling and F. A. Sukochev, *Operator Lipschitz estimates in the unitary setting*, Proc. Amer. Math. Soc. 144 (2016), 1053–1057.

[3] N. Azamov, A. Carey, P. Dodds and F. Sukochev, *Operator integrals, spectral shift, and spectral flow*, Canad. J. Math. 61 (2009), 241–263.

[4] M. Birman and M. Solomyak, *Double Stieltjes operator integrals III*, in: Problems Math. Phys. 6, Izdat. Leningrad. Univ., 1973, 27–53 (in Russian).

[5] A. Chattopadhyay, C. Coine, S. Giri and C. Pradhan, *Higher order $\mathcal{S}^p$-differentiability: The unitary case*, J. Spectr. Theory 15 (2025), 195–222.

[6] C. Coine, *Perturbation theory and higher order $\mathcal{S}^p$-differentiability of operator functions*, Rev. Mat. Iberoamer. 38 (2022), 189–221.

[7] C. Coine, C. Le Merdy, A. Skripka and F. Sukochev, *Higher order $\mathcal{S}^2$-differentiability and application to Koplienko trace formula*, J. Funct. Anal. 276 (2019), 3170–3204.

[8] C. Coine, C. Le Merdy and F. Sukochev, *When do triple operator integrals take value in the trace class?* Ann. Inst. Fourier (Grenoble) 71 (2021), 1393–1448.

[9] J. B. Conway, *A Course in Operator Theory*, Grad. Stud. Math. 21, Amer. Math. Soc., Providence, RI, 2000.

[10] J. H. Curtiss, *Limits and bounds for divided differences on a Jordan curve in the complex domain*, Pacific J. Math. 12 (1942) 1217–1233.
[11] Yu. L. Daletskii and S. G. Krein, *Integration and differentiation of functions of Hermitian operators and application to the theory of perturbations*, Voronezh. Gos. Univ. Trudy Sem. Funktsional. Anal. 1956, no. 1, 81–105 (in Russian).
[12] B. De Pagter and F. Sukochev, *Differentiation of operator functions in non-commutative L_p-spaces*, J. Funct. Anal. 212 (2004), 28–75.
[13] Yu. B. Farforovskaya, *An estimate of the norm of $|f(B) - f(A)|$ for selfadjoint operators A and B*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 56 (1976), 143–162 (in Russian).
[14] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol. I, Elementary Theory*, Pure Appl. Math. 100, Academic Press, New York, 1983.
[15] E. Kissin, D. Potapov, V. Shulman and F. Sukochev, *Operator smoothness in Schatten norms for functions of several variables: Lipschitz conditions, differentiability and unbounded derivations*, Proc. London Math. Soc. 108 (2014), 327–349.
[16] E. Kissin and V. Shulman, *Classes of operator-smooth functions. II. Operator-differentiable functions*, Integral Equations Operator Theory 49 (2004), 165–210.
[17] C. Le Merdy and A. Skripka, *Higher order differentiability of operator functions in Schatten norms*, J. Inst. Math. Jussieu 19 (2020), 1993–2016.
[18] H. Neidhardt, *Spectral shift function and Hilbert–Schmidt perturbation: extensions of some work of L. S. Koplienko*, Math. Nachr. 138 (1988), 7–25.
[19] G. K. Pedersen, *Operator differentiable functions*, Publ. Res. Inst. Math. Sci. 36 (2000), 139–157.
[20] V. V. Peller, *Multiple operator integrals and higher operator derivatives*, J. Funct. Anal. 233 (2006), 515–544.
[21] D. Potapov, A. Skripka and F. Sukochev, *Spectral shift function of higher order*, Invent. Math. 193 (2013), 501–538.
[22] D. Potapov, A. Skripka and F. Sukochev, *Higher-order spectral shift for contractions*, Proc. London Math. Soc. (3) 108 (2014), 327–349.
[23] D. Potapov, A. Skripka and F. Sukochev, *Trace formulas for resolvent comparable operators*, Adv. Math. 272 (2015), 630–651.
[24] D. Potapov, A. Skripka and F. Sukochev, *Functions of unitary operators: derivatives and trace formulas*, J. Funct. Anal. 270 (2016), 2048–2072.
[25] D. Potapov, A. Skripka, F. Sukochev and A. Tomskova, *Multilinear Schur multipliers and Schatten properties of operator Taylor remainders*, Adv. Math. 320 (2017), 1063–1098.
[26] A. Skripka, *Estimates and trace formulas for unitary and resolvent comparable perturbations*, Adv. Math. 311 (2017), 481–509.
[27] A. Skripka and A. Tomskova, *Multilinear Operator Integrals. Theory and Applications*, Lecture Notes in Math. 2250, Springer, Cham, 2019.
[28] V. V. Sten'kin, *Multiple operator integrals*, Izv. Vyssh. Uchebn. Zaved. Matematika 1977, no. 4 (179), 102–115 (in Russian); English transl.: Soviet Math. (Iz. VUZ) 21 (1977), no. 4, 88–99.

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