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Continuous linear and bilinear Schur multipliers and applications to perturbation theory

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Introduction

1 Résumé de la thèse

Le travail de cette thèse a été dans un premier temps motivé par la résolution du problème de Peller concernant la formule de trace de Koplienko-Neidhardt. Celui-ci est en lien avec les perturbations du second ordre pour le calcul fonctionnel. En effet, le problème était de déterminer, pour une fonction $f \in C^2(\mathbb{R})$ dont la dérivée seconde est bornée, et pour deux opérateurs autoadjoints A et K sur un espace de Hilbert séparable $\mathcal{H}$ tels que $K \in \mathcal{S}^2(\mathcal{H})$ est un opérateur de Hilbert-Schmidt, si l'opérateur

$$f(A+K) - f(A) - \frac{d}{dt}\Big(f(A+tK)\Big)\Big|_{t=0} \tag{1}$$

appartient à l'espace $\mathcal{S}^1(\mathcal{H})$ des opérateurs à trace.
Cette question a été soulevée par V. Peller dans [Pel05], où il a également conjecturé que la réponse à cette question était négative.
Pour résoudre ce problème, il est important de comprendre tout d'abord dans quels cas l'opérateur (1) est bien défini. Lorsque A est borné ou quand f a une dérivée bornée, l'opérateur est bien défini et appartient à $\mathcal{S}^2(\mathcal{H})$. Sinon, le sens de (1) n'est pas clair, mis à part dans certains cas particuliers. Peller a par exemple défini dans [Pel05] l'opérateur (1) par approximation lorsque f appartient à la classe de Besov $B^2_{\infty 1}(\mathbb{R})$ et a alors montré que la question précédente était positive pour de telles fonctions. Deuxièmement, il est commode d'exprimer différemment (1). Il s'avère que ceci peut être fait au moyen des 'Opérateurs intégraux triple'. La théorie des opérateurs intégraux multiple a été initiée par Birman et Solomyak, dans une série de trois articles (voir [BS66; BS67; BS73]). Dans les 20 dernières années, de nombreux développements ont été obtenus par V. Peller, F. Sukochev, et leurs co-auteurs. Ces objets jouent un rôle majeur dans la théorie de la perturbation. Un opérateur intégral double est un opérateur de la forme

$$\Gamma^{A,B}(\phi) : \mathcal{S}^2(\mathcal{H}) \to \mathcal{S}^2(\mathcal{H})$$

associé à deux opérateurs normaux A et B sur $\mathcal{H}$ et à une fonction borélienne ϕ bornée sur le produit des spectres de A et B. Un des premiers résultats majeurs est la formule

$$f(A+K) - f(A) = \left[\Gamma^{A+K,A}(f^{[1]})\right](K) \tag{2}$$

où $K \in \mathcal{S}^2(\mathcal{H})$, f est une fonction Lipschitzienne et $f^{[1]}$ est la différence divisée d'ordre 1 de f. Parmi les applications importantes de cette formule, nous pouvons citer l'étude des 'fonctions Lipschitz-opérateurs', c'est-à-dire l'espace des fonctions lipschitziennes sur $\mathbb{R}$ qui ont une propriété de Lipschitz pour le calcul fonctionnel des opérateurs autoadjoints. L'un des résultats importants de ce sujet a été obtenu par D. Potapov et F. Sukochev qui ont établi que toute fonction Lipschitzienne était opérateur-Lipschitz sur

les classes de Schatten réflexives $\mathcal{S}^p, 1 < p < \infty$ (voir [PS11]), où l'utilisation de (2) a été fondamentale. Ce résultat est faux dans le $p = 1$ et $p = \infty$ et un contre-exemple a été construit e.g. dans [Far72].
Les opérateurs intégraux triple sont des applications bilinéaires définies sur $\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H})$ et à valeurs dans $\mathcal{S}^2(\mathcal{H})$. Avec le même type de formule, l'opérateur (1) peut être obtenu comme un certain opérateur intégral triple. Il s'avère que les opérateurs intégraux double et triple peuvent être vus comme des multiplicateurs de Schur linéaires et bilinéaires continus. En effet, lorsque l'espace de Hilbert $\mathcal{H}$ est de dimension finie, l'action des opérateurs intégraux double et triple est identique à celle des multiplicateurs de Schur classiques, qui ont été intensément étudiés.
D'après la discussion qui précède, une idée pour résoudre le problème de Peller est de comprendre dans quels cas un opérateur intégral triple est à valeurs dans $\mathcal{S}^1(\mathcal{H})$. La première étape est de comprendre ce phénomène dans le cas discret, c'est-à-dire dans quels cas un multiplicateur de Schur bilinéaire est à valeurs dans $\mathcal{S}^1$. Il se trouve que la norme $\mathcal{S}^1$ de telles applications peut être calculée à l'aide de normes de multiplicateurs de Schur linéaires sur $\mathcal{B}(\mathbb{C}^n)$. Ces objets sont bien connus et il existe une description des multiplicateurs de Schur linéaires sur $\mathcal{B}(\ell_2)$ (voir par exemple [Pis96, Théorème 5.1]). Ce lien inattendu entre le problème de Peller et les multiplicateurs de Schur a été le point de départ pour la résolution du problème, et plus précisément, pour la construction d'un contre-exemple.

Cette thèse s'organise de la façon suivante.

Dans le Chapitre 1, nous définissons différentes notions qui joueront un rôle important dans cette thèse, même si beaucoup d'entre elles n'apparaissent pas explicitement dans l'énoncé des résultats principaux. Nous utiliserons souvent les produits tensoriels comme des outils pour les démonstrations, et en particulier, l'identification du dual de certains produits tensoriels de deux espaces de Banach est en général la clé pour d'importants résultats. Comme nous l'avons déjà mentionné dans la première partie de l'introduction, de nombreuses questions de cette thèse seront formulées avec des classes de Schatten. Nous rappellerons leur définition et quelques propriétés de ces espaces importants. Les deux dernières sections concerneront les espaces L^p_σ qui apparaissent comme les espaces duaux des espaces de Bochner. En particulier, la section 1.4 concerne d'importants résultats de factorisation pour les espaces L^∞_σ à valeurs dans l'espace des opérateurs factorisables par un espace de Hilbert. Ces résultats ont été obtenus en collaboration avec C. Le Merdy et F. Sukochev et apparaissent dans l'article [CMS17].

Dans le Chapitre 2, nous nous intéresserons aux multiplicateurs de Schur linéaires. Le résultat principal les concernant est une caractérisation des multiplicateurs de Schur sur $\mathcal{B}(\ell_2)$ par Grothendieck. Notre but a été de généraliser ce résultat et de caractériser les multiplicateurs de Schur sur $\mathcal{B}(\ell_p, \ell_q)$. Nous avons pu le faire dans le $q \leq p$ et avons obtenu un résultat similaire à celui de Grothendieck. Comme nous l'avons expliqué précédemment, les objets apparaissant dans cette thèse sont des multiplicateurs de Schur continus. Ainsi, nous définirons plus généralement les multiplicateurs de Schur continus sur $\mathcal{B}(L^p, L^q)$. Nous verrons que pour les comprendre, il suffit de comprendre les multiplicateurs de Schur classiques. Nous terminerons ce chapitre avec de nouveaux résultats concernant les relations d'inclusion entre les espaces de multiplicateurs de Schur. Les résultats de cette section apparaissent dans l'article [Coi17].

Nous étudierons dans le Chapitre 3 les multiplicateurs de Schur bilinéaires, dans le cas classique ainsi que dans le cas continu. Après avoir rappelé leur définition, notre but sera d'étudier la bornitude dans $\mathcal{S}^1$ de tels opérateurs. Les résultats principaux de cette partie sont des caractérisations des multiplicateurs de Schur bilinéaires à valeurs dans $\mathcal{S}^1$ à l'aide de multiplicateurs de Schur linéaires. Ces résultats seront la première mais également l'une des principales étapes pour comprendre et résoudre le problème de Peller.

Le Chapitre 4 est dédié à divers résultats sur les opérateurs intégraux multiple. Nous donnerons tout d'abord une définition de ces opérateurs par dualité, ce qui permettra d'obtenir une définition plus générale que celles introduites auparavant. Comme nous l'avons déjà dit, ces objets peuvent être vus comme des multiplicateurs de Schur multilinéaires continus. Ainsi, en utilisant les résultats obtenus dans les chapitres précédents, nous serons en mesure de caractériser les opérateurs intégraux triple à valeurs dans l'espace des opérateurs à trace. Enfin, dans une dernière section, nous donnerons une condition nécessaire et suffisante pour qu'un opérateur intégral triple définisse une application complètement bornée de $S^\infty(\mathcal{H}) \otimes S^\infty(\mathcal{H})$ muni du produit de Haagerup à valeurs dans $S^\infty(\mathcal{H})$. Ceci généralise au cas des opérateurs intégraux un résultat obtenu dans [KJT09] dans le cadre des multiplicateurs de Schur multilinéaires continus. Les résultats des sections 4.1 et 4.3 ont été obtenus en collaboration avec C. Le Merdy et F. Sukochev et l'article [CMS17] a été écrit à ce sujet.

Enfin, nous résoudrons dans le Chapitre 5 le problème de Peller. Nous avons mentionné le cas autoadjoint mais un problème similaire peut être formulé dans le cas unitaire. Ces deux problèmes seront résolus en utilisant les mêmes idées. Le premier outil sera la connexion entre les problèmes de Peller et les opérateurs intégraux triple. Pour ce faire, nous étudierons le lien entre opérateurs intégraux multiple et théorie de la perturbation pour les opérateurs autoadjoints. En particulier, nous donnerons une formule pour la dérivée $n-$ième des applications de la forme

$$t \in \mathbb{R} \mapsto f(A + tK) - f(A)$$

où A et K sont des opérateurs autoadjoints avec K un opérateur de Hilbert-Schmidt. Ce résultat est une généralisation de la formule (2) et nous obtiendrons alors une formule de Taylor à l'ordre n pour les opérateurs autoadjoints. En particulier, l'opérateur (1) apparaîtra comme un certain opérateur intégral triple. Le second outil sera le calcul de la norme $\mathcal{S}^1$ pour un multiplicateur de Schur bilinéaire à l'aide de multiplicateurs de Schur linéaires ce qui nous permettra d'exploiter un contre-exemple dû à E. B. Davies concernant le comportement de l'application valeur absolue sur les espaces $\mathcal{S}^1(\mathbb{C}^n), n \in \mathbb{N}$. En utilisant des estimations de normes dans le cas fini dimensionnel, nous construirons deux opérateurs A et K comme sommes directes d'opérateurs de rang fini tels que l'opérateur (1) n'appartient pas à $\mathcal{S}^1$, où f sera une fonction bien choisie. Les résultats de la section 5.2 ont été obtenus en collaboration avec C. Le Merdy et A. Skripka. Les résultats des sections 5.3 et 5.4 ainsi que ceux de la sous-section 3.3.2 ont quant à eux été obtenus en collaboration avec C. Le Merdy, F. Sukochev, D. Potapov et A. Tomskova et les deux articles [CMPST16a; CMPST16b] ont été publiés les concernant.

2 Summary of the thesis

The work in this thesis was first motivated by the resolution of Peller's problem concerning Koplienko-Neidhardt trace formulae. It is related to perturbations of second order for functional calculus. Indeed, the problem was to determine, for a function $f \in C^2(\mathbb{R})$ with bounded second derivative, and for two selfadjoint operators A, K acting on a separable Hilbert space $\mathcal{H}$ such that $K \in \mathcal{S}^2(\mathcal{H})$ is a Hilbert-Schmidt operator, whether the operator

$$f(A+K) - f(A) - \frac{d}{dt}\Big(f(A+tK)\Big)\Big|_{t=0} \tag{3}$$

is in the space $\mathcal{S}^1(\mathcal{H})$ of trace class operators.
This question was stated by V. Peller in [Pel05], where he also suggested that this question should have a negative answer.
To solve this problem, it is first important to understand in which cases the operator in (3) is well-defined. When A is a bounded operator or when f has a bounded derivative, the operator is well-defined and is an element of $\mathcal{S}^2(\mathcal{H})$. Otherwise, the meaning of (3) is not clear, except in certain particular cases. For instance, Peller proved in [Pel05] that when f belongs to the Besov class $B^2_{\infty 1}$ the operator (3) can be defined by approximation and that in this case, the question stated above holds true. Secondly, it is convenient to express (3) differently. It turns out that this can be done by means of the so-called triple operator integrals. The theory of multiple operator integrals started with Birman and Solomyak, in a series of three papers (see [BS66; BS67; BS73]). In the last 20 years, outstanding developments have been made by V. Peller, F. Sukochev, and their co-authors. They play a major role in perturbation theory. A double operator integral is an operator of the form

$$\Gamma^{A,B}(\phi) : \mathcal{S}^2(\mathcal{H}) \to \mathcal{S}^2(\mathcal{H})$$

associated to normal operators A, B on $\mathcal{H}$ and a Borel function ϕ bounded on the product of the two spectra $\sigma(A) \times \sigma(B)$ of A and B. One of the early results is the formula

$$f(A+K) - f(A) = \big[\Gamma^{A+K,A}(f^{[1]})\big](K) \tag{4}$$

where $K \in \mathcal{S}^2(\mathcal{H})$, f is a Lipschitz function and $f^{[1]}$ is the divided difference of first order of f. Among the important applications of such formula, we can mention the study of 'Operator-Lipschitz function', that is, the space of Lipschitz functions on $\mathbb{R}$ which have a Lipschitz property for functional calculus of selfadjoint operators. One the very important results in this direction was obtained by D. Potapov and F. Sukochev who established that any Lipschitz function is Lipschitz operator on the reflexive Schatten classes $\mathcal{S}^p, 1 < p < \infty$ (see [PS11]), and where the use of (4) was fundamental. This result does not hold true in the case $p = 1$ and $p = \infty$ and a counterexample was built in [Far72].
Triple operator integrals are bilinear mappings defined on $\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H})$ and valued

in $\mathcal{S}^2(\mathcal{H})$. With the same kind of formula, the operator (3) can be obtained as a certain triple operator integral. It turns out that double and triple operator integrals can be understood as continuous linear and bilinear Schur multipliers, respectively. Indeed, for a finite dimensional Hilbert space $\mathcal{H}$, double and triple operator integrals are nothing but the classical linear and bilinear Schur multipliers which have been intensively studied.

According to the discussion above, an idea to solve Peller's problem is to determine in which case a triple operator integral is actually valued in $\mathcal{S}^1(\mathcal{H})$. The first step is to understand that in the finite dimensional case, that is, in which case a bilinear Schur multiplier is valued in $\mathcal{S}^1$. It turns out that the $\mathcal{S}^1$-norm of such mappings can be computed thanks the the norms of a family of linear Schur multipliers on $\mathcal{B}(\mathbb{C}^n)$. These objects are well-known and there is description of linear Schur multipliers on $\mathcal{B}(\ell_2)$ (see e.g. [Pis96, Theorem 5.1]). This unexpected connection between Peller's problem and linear Schur multiplier was the starting point for the resolution of the problem, and more precisely, for the construction of a counter-example.

This thesis is organized as follow.

In Chapter 1, we define several notions that will play an important role in this thesis, even if many of them do not appear explicitly in the statements of the main results. We will often use tensor products as a tool, and in particular, the identification of the dual of certain tensor products of two Banach spaces is usually the key for many important results. As we already mentionned them in the first part of this introduction, many questions in this thesis will be stated with Schatten classes. We will recall their definition and several properties of those important spaces. The last two sections will deal with the L^p_σ-spaces which appear as the dual of Bochner spaces. In particular, we prove in Section 1.4 important factorization properties for L^∞_σ-spaces valued in the space of operators that can be factorized by a Hilbert space. Those results have been obtained in collaboration with C. Le Merdy and F. Sukochev and appear in [CMS17].

In Chapter 2, we will be interested in linear Schur multipliers. The main result concerning them is a characterization of Schur multipliers on $\mathcal{B}(\ell_2)$ by Grothendieck. Our aim was to generalize this result in order to obtain a characterization of Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$. It turns out that we could manage the case $q \leq p$ and obtain a statement similar to the one of Grothendieck. As we explained before, the objects appearing in this thesis are continuous Schur multipliers. Therefore, we will define more generally continuous Schur multipliers on $\mathcal{B}(L^p, L^q)$. We will see that to understand them, it is enough to understand classical Schur multipliers. We will finish this chapter with several new results about the inclusions between the spaces of Schur multipliers. The article [Coi17] has been written concerning the results of this chapter.

In Chapter 3, we study bilinear Schur multipliers, in the classical and in the continuous case. After recalling their definitions, our concern will be the $\mathcal{S}^1$-boundedness of such operators. The main results are characterizations of bilinear Schur multipliers valued in $\mathcal{S}^1$ by the use of linear Schur multipliers. Those results will be the first and the key step to understand and solve Peller's problem.

Chapter 4 is dedicated to various results about multilinear operator integrals. We first give a definition of those operators by duality, which allows us to have a more general definition than the ones introduced before. As we already said, those objects can be understood as a kind of continuous multilinear Schur multipliers. Thus, using our preceding results, we will be able to characterize triple operator integrals that are valued in the trace class operators. In a last section we will give a necessary and sufficient condition for a triple operator integral to define a completely bounded map from $S^\infty(\mathcal{H}) \otimes S^\infty(\mathcal{H})$ equipped with the Haagerup tensor product into $S^\infty(\mathcal{H})$. This generalizes a result obtained in [KJT09] in the setting of continuous multilinear Schur multipliers. The results of Sections 4.1 and 4.3 have been obtained in collaboration with C. Le Merdy and F. Sukochev and the paper [CMS17] has been written about them.

Finally, Chapter 5 is the resolution of Peller's problem. We mentionned the selfadjoint case but a similar problem can be stated in the unitary case. We will solve both problems using the same ideas. The first tool will be the connection between Peller's problems and triple operator integrals. To do so, we will study the connection between multilinear operator integrals and perturbation theory. In particular, we give a formula for the $n-th$ derivative of a map of the form

$$t \in \mathbb{R} \mapsto f(A+tK) - f(A)$$

where A and K are selfadjoint operators with K a Hilbert-Schmidt operator. This result will generalize Formula (4) and we will obtain a Taylor formula at the order n for selfadjoint operators. In particular, the operator (3) will appear as a certain triple operator integral. The second tool will be the computation of the $\mathcal{S}^1$-norm of a bilinear Schur multiplier by means of linear Schur multipliers which will allow us to use a counterexample of E. B. Davies concerning the behavior of the absolute value mapping on the spaces $\mathcal{S}^1(\mathbb{C}^n), n \in \mathbb{N}$. By using norm estimates in the finite-dimensional case, we will construct two operators A and K as a direct sum of finite rank operators such that the operator (3) does not belong to $\mathcal{S}^1$, where f is a well chosen function. The results of Section 5.2 have been obtained in collaboration with C. Le Merdy A. Skripka. The results of Sections 5.3 and 5.4 as well as those of Subsection 3.3.2 have been obtained in collaboration with C. Le Merdy, D. Potapov, F. Sukochev and A. Tomskova and two papers [CMPST16a; CMPST16b] have been published concerning them.

3 Notations

We give here a few notations that will be used throughout this thesis. The notations that are used later but not mentioned here are either standard or they will be given when needed.

- $\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$ will denote the unit circle of the complex plane.
- Let $1 \leq p < +\infty$. We define

$$\ell_p = \left\{ x = (x_n)_{n=1}^{+\infty} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

equipped with the norm $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$.

- If $p = +\infty$, let
$$\ell_\infty = \left\{ x = (x_n)_{n=1}^{+\infty} : \sup_n |x_n| < \infty \right\}$$
equipped with the norm $\|x\|_\infty = \sup_n |x_n|$.

- If $n \in \mathbb{N}$, we denote by ℓ_p^n the $n-$dimensional versions of the spaces introduced before.

- For a Hilbert space $\mathcal{H}$, let $\overline{\mathcal{H}}$ denote its conjugate space.

- The Hilbertian direct sum of any sequence $(\mathcal{H}_n)_{n\geq 1}$ of Hilbert space will be denoted by
$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n.$$
In this case, if for all $n \geq 1$, A_n is a bounded operator acting on $\mathcal{H}_n$, we will denote by $A = \bigoplus_{n=1}^{\infty} A_n$ the operator defined on the domain
$$D(A) = \left\{ \{h_n\}_{n=1}^{\infty} \in \mathcal{H} \ : \ \sum_{n=1}^{\infty} \|A_n(h_n)\|^2 \ < \infty \right\},$$
by setting $A(h) = \{A_n(h_n)\}_{n=1}^{\infty}$ for any $h = \{h_n\}_{n=1}^{\infty}$ in $D(A)$.
For two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we will denote by $\mathcal{H} \overset{2}{\oplus} \mathcal{K}$ their Hilbertian direct sum.

- Whenever Σ is a set and $V \subset \Sigma$ is a subset we let $\chi_V : \Sigma \to \{0,1\}$ denote the characteristic function of V.

Let X and Y be two Banach spaces.

- For $1 \leq p \leq \infty$ and a measure space (Ω, μ) we denote by $L^p(\Omega; X)$ the Bochner space of $p-$ integrable (classes) of functions $f : \Omega \to X$.
When $X = \mathbb{C}$, we simply write $L^p(\Omega)$.

- $\mathcal{B}(X, Y)$ is the Banach space of bounded linear operators $T : X \to Y$ equipped with the operator norm $\|.\|$ defined by
$$\|T\| = \sup_{x \in X, \|x\| \leq 1} \|Tx\|.$$
When $Y = \mathbb{C}$, we write X^* for the dual space of X.

- Let Z be a third Banach space. We let $\mathcal{B}_2(X \times Y, Z)$ be the Banach space of all bounded bilinear operators $T\colon X \times Y \to Z$, equipped with
$$\|T\| = \sup\big\{ \|T(x,y)\| \ : \ x \in X, \ y \in Y, \ \|x\| \leq 1, \ \|y\| \leq 1 \big\}.$$

- We write $X_1 \otimes \cdots \otimes X_n$ for the tensor product of n Banach spaces $X_1, \ldots, X_n$. When $X_i \subset L^\infty(\Omega_i)$ for some measure spaces (Ω_i, μ_i), we will often identify an element $f = f_1 \otimes \cdots \otimes f_n \in X_1 \otimes \cdots \otimes X_n$ with an element of $L^\infty(\Omega_1 \times \ldots \times \Omega_n)$ as follows:

$$\forall t = (t_1, \ldots, t_n) \in \Omega_1 \times \ldots \times \Omega_n, f(t) = f_1(t_1) \ldots f_n(t_n).$$

Let $E \subset \mathcal{B}(H)$ and $F \subset \mathcal{B}(K)$ be two operator spaces.

- For $n, m \in \mathbb{N}^*$, let $M_{n,m}(E)$ be the space of $n \times m$-matrices with entries in E. For $r \in \mathbb{N}^*$, denote by $\ell_2^r(\mathcal{H})$ the space $\bigoplus_{k=1}^r \mathcal{H}$. We have an identification

$$M_{n,m}(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\ell_2^m(\mathcal{H}), \ell_2^n(\mathcal{H})).$$

  Hence, we may equip $M_{n,m}(E)$ with the norm induced by the inclusion

$$M_{n,m}(E) \subset M_{n,m}(\mathcal{B}(\mathcal{H})).$$

- Let $u : E \to F$ be a linear map. For $n \in \mathbb{N}^*$, write $M_{n,n}(E) := M_n(E)$. We consider the mapping $u_n : M_n(E) \to M_n(F)$ defined, for $x = [e_{ij}]_{1 \le i,j \le n} \in M_n(E)$ by

$$u_n(x) = [u(e_{ij})]_{1 \le i,j \le n}.$$

  We say that u is completely bounded if

$$\|u\|_{\text{cb}} := \sup_n \|u_n\| < \infty,$$

  and we denote by $CB(E, F)$ the Banach space of completely bounded maps from E into F equipped with the c.b. norm.
  If for any n, u_n is contractive (respectively positive, resp. an isometry), we say that u is completely contractive (resp. completely positive, resp. a complete isometry).

- If $\mathcal{H}$ is a Hilbert space, we denote by $\mathcal{H}_c = \mathcal{B}(\mathbb{C}, \mathcal{H})$ its column structure and by $\mathcal{H}_r = \mathcal{B}(\overline{\mathcal{H}}, \mathbb{C})$ its row structure. We refer e.g. to [ER00, Section 3.4] for further informations.

- In Chapter 4, the L^1-spaces will be equipped with their maximal operator space structure (Max) for which we refer to [Pis03, Chapter 3].

Chapter 1

Preliminaries

In this first chapter, we give some preliminary results that we will use all along the thesis. First, we will give some background on the norms of tensor products. We will define several tensor norms and identify, for two Banach spaces X and Y, the dual of $X \otimes Y$ equipped with those norms. Then, we will give a few properties of Schatten classes. In particular, the spaces of Hilbert-Schmidt operators and trace class operators will play fundamental roles, as they appear in many important definitions and results presented here. In a third section, we will define the L^p_σ-spaces, which are a dual version of Bochner spaces. Finally, the last section of this chapter is of independent interest: it describes the elements of L^p_σ-spaces valued in certain tensor products. This section will be fundamental to give a precise and concrete meaning for important results in Chapters 3 and 4.

1.1 Tensor products

We give a brief summary of tensor product formulas to be used in the sequel.

1.1.1 Projective and injective tensor product

Let E and F be Banach spaces.

• The projective norm:

If $z \in E \otimes F$, the *projective tensor norm of* z is defined by

$$\|z\|_\wedge := \inf \left\{ \sum \|x_i\| \|y_i\| \right\},$$

where the infimum runs over all finite families $(x_i)_i$ in E and $(y_i)_i$ in F such that

$$z = \sum_i x_i \otimes y_i.$$

The completion $E \overset{\wedge}{\otimes} F$ of $(E \otimes F, \| \ \|_\wedge)$ is called the projective tensor product of E and F.

Let G be a Banach space. To any $T \in \mathcal{B}_2(E \times F, G)$, one can associate a linear map $\widetilde{T}\colon E \otimes F \to G$ by the formula

$$\widetilde{T}(x \otimes y) = T(x, y), \qquad x \in E,\ y \in F.$$

Then $\widetilde{T}$ is bounded on $(E \otimes F, \| \, \|_\wedge)$, with $\|\widetilde{T}\| = \|T\|$, and hence the mapping $T \mapsto \widetilde{T}$ gives rise to an isometric identification

$$\mathcal{B}_2(E \times F, G) = \mathcal{B}(E \overset{\wedge}{\otimes} F, G). \tag{1.1}$$

In the case $G = \mathbb{C}$, this implies that the mapping taking any functional $\omega\colon E \otimes F \to \mathbb{C}$ to the operator $u\colon E \to F^*$ defined by $\langle u(x), y\rangle = \omega(x \otimes y)$ for any $x \in E, y \in F$, induces an isometric identification

$$(E \overset{\wedge}{\otimes} F)^* = \mathcal{B}(E, F^*). \tag{1.2}$$

We refer to [DU79, Chapter 8, Theorem 1 and Corollary 2] for these classical facts.

Let (Ω, μ) be a σ-finite measure space and let $L^1(\Omega; F)$ denote the Bochner space of integrable functions from Ω into F. By [DU79, Chapter 8, Example 10], the natural embedding $L^1(\Omega) \otimes F \subset L^1(\Omega; F)$ extends to an isometric isomorphism

$$L^1(\Omega; F) = L^1(\Omega) \overset{\wedge}{\otimes} F. \tag{1.3}$$

By (1.2), this implies

$$L^1(\Omega; F)^* = \mathcal{B}(L^1(\Omega), F^*). \tag{1.4}$$

Assume now that $Y = L^1(\Omega')$ where (Ω', μ') is a σ-finite measure space. Then, an application of Fubini's theorem gives

$$L^1(\Omega, L^1(\Omega')) = L^1(\Omega \times \Omega').$$

Using equality (1.4), we obtain an isometric w^*-homeomorphic identification

$$\mathcal{B}(L^1(\Omega), L^\infty(\Omega')) = L^\infty(\Omega \times \Omega'), \tag{1.5}$$

and the correspondance is given by

$$\begin{array}{ccc} L^\infty(\Omega \times \Omega') & \longrightarrow & \mathcal{B}(L^1(\Omega), L^\infty(\Omega')). \\ \psi & \longmapsto & \left[f \in L^1(\Omega) \mapsto \displaystyle\int_\Omega f(t)\psi(t, \cdot)\mathrm{d}\mu(t) \right] \end{array}$$

For $\psi \in L^\infty(\Omega \times \Omega')$, denote by u_ψ the corresponding element of $\mathcal{B}(L^1(\Omega), L^\infty(\Omega'))$.

- **The injective norm:**

If $z = \sum_i x_i \otimes y_i \in X \otimes Y$, $x^* \in X^*$ and $y^* \in Y^*$, we write

$$\langle z, x^* \otimes y^* \rangle = \sum_i x^*(x_i) y^*(y_i).$$

Then, the *injective tensor norm of* $z \in X \otimes Y$ is given by

$$\|z\|_\vee = \sup_{\|x^*\|\le 1, \|y^*\|\le 1} |\langle z, x^* \otimes y^* \rangle|.$$

The completion $X \overset{\vee}{\otimes} Y$ of $(X \otimes Y, \|.\|_\vee)$ is called the injective tensor product of X and Y.

In this thesis, we will often identify $X^* \otimes Y$ with the finite rank operators from X into Y as follow. If $u = \sum_i x_i^* \otimes y_i \in X^* \otimes Y$, we define $\tilde{u} : X \to Y$ by

$$\tilde{u}(x) = \sum_i x_i^*(x) y_i, \forall x \in X. \tag{1.6}$$

Then, it is easy to check that $\|u\|_\vee = \|\tilde{u}\|_{\mathcal{B}(X,Y)}$.

Moreover, if Y has the approximation property (see e.g. [DFS08] for the definition), [DFS08, Theorem 1.4.21] gives the isometric identification

$$X^* \overset{\vee}{\otimes} Y = \mathcal{K}(X, Y)$$

where $\mathcal{K}(X, Y)$ denotes the space of compact operators from X into Y.

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. Let $1 \le p < \infty$ and $1 \le q \le \infty$. Then $L^q(\Omega_2)$ has the approximation property so that we have

$$L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) = \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)). \tag{1.7}$$

Finally, if we assume that $1 < p, q < +\infty$, then by [DFS03, Theorem 2.5] and (1.2),

$$(L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2))^{**} = (L^p(\Omega_1) \overset{\wedge}{\otimes} L^{q'}(\Omega_2))^* = \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2)). \tag{1.8}$$

1.1.2 Lapresté norms

Let $s \in [1, \infty]$. If $x_1, x_2, \ldots, x_n \in X$, we define

$$w_s(x_i, X) := \sup_{x^* \in X^*, \|x^*\| \le 1} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^s \right)^{1/s}.$$

Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} \ge 1$ and take $r \in [1, \infty]$ such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Denote by p' and q' the conjugate of p and q. For $z \in X \otimes Y$, we define

$$\alpha_{p,q}(z) = \inf \left\{ \|(\lambda_i)_i\|_r w_{q'}(x_i, X) w_{p'}(y_i, Y) \mid z = \sum_{i=1}^n \lambda_i x_i \otimes y_i \right\}.$$

Then $\alpha_{p,q}$ is a norm on $X \otimes Y$ and we denote by $X \otimes_{\alpha_{p,q}} Y$ its completion (see e.g. [DF93, Proposition 12.5]).

1.1.3 Haagerup tensor product

Let $E \subset \mathcal{B}(\mathcal{H})$ and $F \subset \mathcal{B}(\mathcal{K})$ be two operators spaces. Let $n \in \mathbb{N}^*$. For $r \in \mathbb{N}^*$ we define the matrix inner product $e \odot f \in M_n(E \otimes F)$ of two elements $e = [e_{ij}] \in M_{n,r}(E)$ and $f = [f_{ij}] \in M_{r,n}(F)$ by

$$e \odot f = \left[\sum_k e_{ik} \otimes f_{kj} \right]_{1 \leq i,j \leq n}.$$

We define, for $u \in M_n(E \otimes F)$,

$$\|u\|_h = \inf \{\|e\| \|f\|\}$$

where the infimum runs over all $r \in \mathbb{N}^*, e = [e_{ij}] \in M_{n,r}(E), f = [f_{ij}] \in M_{r,n}(F)$ such that $u = e \odot f$. By [ER00, Lemma 9.1.1], such factorization of u exists.

Note that for $x \in E \otimes F$ we have

$$\|x\|_h = \inf \left\{ \left\| \sum_i a_i a_i^* \right\|^{1/2} \left\| \sum_i b_i^* b_i \right\|^{1/2}, \ x = \sum_i a_i \otimes b_i \right\}.$$

Then $\|.\|_h$ satisfies the axioms of Ruan's theorem (see e.g. [Pis03, Section 2.2]), hence, after completion, we obtain an operator space denoted by $E \overset{h}{\otimes} F$.

A first property of the Haagerup tensor product is its associativity. Indeed, if G is another operator space, we have, by [ER00, Proposition 9.2.7], a complete isometry

$$(E \overset{h}{\otimes} F) \overset{h}{\otimes} G = E \overset{h}{\otimes} (F \overset{h}{\otimes} G).$$

See also [Pis03, Chapter 5] for a definition of $E_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} E_N$ for N operator spaces $E_1, \ldots, E_N$.

We give now a few properties of the Haagerup tensor product that we will use in Chapter 4.

Theorem 1.1. *[ER00, Theorem 9.4.3] Let E_1 and E_2 be operator spaces and let H_0 and H_2 be Hilbert spaces. A linear mapping*

$$u : E_1 \overset{h}{\otimes} E_2 \to \mathcal{B}(H_2, H_0)$$

is completely bounded if and only if there exist a Hilbert space H_1 and completely bounded mappings $\phi_i : E_i \to \mathcal{B}(H_i, H_{i-1})$ $(i = 1, 2)$ such that

$$u(x_1 \otimes x_2) = \phi_1(x_1)\phi_2(x_2).$$

In this case we can choose ϕ_i such that

$$\|u\|_{cb} = \|\phi_1\|_{cb}\|\phi_2\|_{cb}.$$

Remark 1.2. *When $H_0 = H_2 = \mathbb{C}$ we can reformulate as follows: a linear functional $u : E_1 \overset{h}{\otimes} E_2 \to \mathbb{C}$ is bounded (and therefore completely bounded) if and only if there exist a Hilbert space H, $\alpha : E_1 \to (H_c)^*$ linear and $\beta : E_2 \to H_c$ antilinear, α and β completely bounded such that*

$$u(x_1, x_2) = \langle \alpha(x_1), \beta(x_2) \rangle .$$

Recall the definition of a quotient map.

Definition 1.3. *Let X and Y be Banach spaces. A map $s : X \to Y$ is a quotient map if s is surjective and for all $y \in Y$ with $\|y\| < 1$, there exists $x \in X$ such that $\|x\| < 1$ and $s(x) = y$. This is equivalent to the fact that the injective map $\hat{s} : X/\ker(s) \to Y$ induced by s is a surjective isometry.*

If $E_1 \subset E_2$ are operator spaces, we equip E_2/E_1 with the quotient operator space structure (see e.g. [Pis03, Section 2.4]). When E and F are operator spaces, a quotient map $u : E \to F$ is said to be a *complete metric surjection* if the associated mapping $\hat{u} : E/\ker(u) \to F$ is a completely isometric isomorphism.

Proposition 1.4. *Let E_1, E_2, F_1, F_2 be operator spaces.*

(i) If $q_i : E_i \to F_i$ is completely bounded, then

$$q_1 \otimes q_2 : E_1 \otimes E_2 \to F_1 \overset{h}{\otimes} F_2$$

defined by $(q_1 \otimes q_2)(e_1 \otimes e_2) = q_1(e_1) \otimes q_2(e_2)$ extends to a completely bounded map

$$q_1 \otimes q_2 : E_1 \overset{h}{\otimes} E_2 \to F_1 \overset{h}{\otimes} F_2.$$

(ii) If $E_i \subset F_i$ completely isometrically, then $E_1 \overset{h}{\otimes} E_2 \subset F_1 \overset{h}{\otimes} F_2$ completely isometrically.

(iii) If $q_i : E_i \to F_i$ is a complete metric surjection, then $q_1 \otimes q_2 : E_1 \overset{h}{\otimes} E_2 \to F_1 \overset{h}{\otimes} F_2$ is also one.

The second property is called the injectivity and the third one the projectivity of the Haagerup tensor product.

Corollary 1.5. *Let X and Y be operator spaces and let $E \subset X, F \subset Y$ be subspaces. Let $p : X \to X/E$ and $q : Y \to Y/F$ be the canonical mappings. They induce a mapping*

$$p \otimes q : X \overset{h}{\otimes} Y \to X/E \overset{h}{\otimes} Y/F.$$

Then

$$\ker(p \otimes q) = \overline{E \otimes Y + X \otimes F}.$$

Proof. Write $N = \overline{E \otimes Y + X \otimes F}$. First note that we easily obtain the first inclusion

$$N \subset \ker(p \otimes q).$$

Therefore, to show the result, it is enough to show that

$$N^{\perp} \subset \ker(p \otimes q)^{\perp}.$$

Let $\sigma : X \overset{h}{\otimes} Y \to \mathbb{C}$ be such that $\sigma_{|N} = 0$. By Remark (1.2), there exist a Hilbert space H, $\alpha : X \to (H_c)^*$ linear and $\beta : Y \to H_c$ antilinear, α and β completely bounded such that

$$\sigma(x, y) = \langle \alpha(x), \beta(y) \rangle, x \in X, y \in Y.$$

Let $K = \overline{\alpha(X)}$ and denote by P_K the orthogonal projection onto K. Then we have, for any x and y,

$$\sigma(x, y) = \langle P_K \alpha(x), \beta(y) \rangle = \langle P_K \alpha(x), P_K \beta(y) \rangle.$$

Thus, by changing α into $P_K \alpha$ and β into $P_K \beta$, we can assume that α has a dense range. Similarly, setting $L = \overline{\beta(Y)}$ and considering P_L, we may assume that β has a dense range.

By assumption, for any $f \in F$ and any $x \in X$, we have

$$0 = \sigma(x, f) = \langle \alpha(x), \beta(f) \rangle.$$

This implies that $\beta_{|F} = 0$. Similarly, we show that $\alpha_{|E} = 0$. Thus, we can consider

$$\widehat{\alpha} : X/E \to (H_c)^* \text{ and } \widehat{\beta} : Y/F \to H_c$$

such that $\alpha = \widehat{\alpha} \circ p$ and $\beta = \widehat{\beta} \circ q$ and where X/E and Y/F are equipped with their quotient structure. This allows to define $\widehat{\sigma} : X/E \overset{h}{\otimes} Y/F \to \mathbb{C}$ by

$$\widehat{\sigma}(s, t) = \left\langle \widehat{\alpha}(s), \widehat{\beta}(t) \right\rangle.$$

Then $\sigma = \widehat{\sigma} \circ (p \otimes q)$, so that $\sigma \in \ker(p \otimes q)^{\perp}$. □

Proposition 1.6. *[ER00, Theorem 9.3.3] Let E be an operator space and let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. For any $T \in CB(E, \mathcal{B}(\mathcal{H}, \mathcal{K}))$ we define a mapping*

$$\sigma_T : \mathcal{K}^* \otimes E \otimes \mathcal{H} \to \mathbb{C}$$

by setting

$$\sigma_T(k^* \otimes e \otimes h) = \langle T(e)h, k \rangle.$$

Then, the mapping $T \mapsto \sigma_T$ induces a complete isometry

$$CB(E, \mathcal{B}(\mathcal{H}, \mathcal{K})) = \left((\mathcal{K}_c)^* \overset{h}{\otimes} E \overset{h}{\otimes} \mathcal{H}_c \right)^*.$$

1.1.4 Dual norm

Let $M \subset X$ and $N \subset Y$ be finite dimensional subspaces (in short, f.d.s). If $u = \sum_{i=1}^{n} x_i \otimes y_i \in M \otimes N$ and $v = \sum_{j=1}^{m} x_j^* \otimes y_j^* \in M^* \otimes N^*$ we set

$$\langle v, u \rangle = \sum_{i,j} \langle x_j^*, x_i \rangle \langle y_j^*, y_i \rangle .$$

Let α be a tensor norm on tensor products of finite dimensional spaces. We define, for $z \in M \otimes N$,

$$\alpha'(z, M, N) = \sup \{ | \langle v, u \rangle | \mid v \in M^* \otimes N^*, \alpha(v) \leq 1 \} .$$

Now, for $z \in X \otimes Y$, we set

$$\alpha'(z, X, Y) = \inf \{ \alpha'(z, M, N) \mid M \subset X, N \subset Y \text{ f.d.s., } z \in M \otimes N \} .$$

α' defines a tensor norm on $X \otimes Y$, called the dual norm of α.

In the sequel, we will write $\alpha'(z)$ instead of $\alpha'(z, X, Y)$ for the norm of an element $z \in X \otimes Y$ when there is no possible confusion.

1.1.5 $(p, q)-$Factorable operators.

If $T \in \mathcal{B}(X, Y^*)$ and $\xi = \sum_i x_i \otimes y_i \in X \otimes Y$, then in accordance with (1.2) we set

$$\langle T, \xi \rangle = \sum_i \langle T(x_i), y_i \rangle .$$

Definition 1.7. *Let $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \geq 1$. Let $T \in \mathcal{B}(X, Y^*)$. We say that $T \in \mathcal{L}_{p,q}(X, Y^*)$ if there exists a constant $C \geq 0$ such that*

$$\forall \xi \in X \otimes Y, \ | \langle T, \xi \rangle | \leq C \alpha'_{p,q}(\xi). \tag{1.9}$$

In this case, we write $L_{p,q}(T) = \inf \{ C \mid C$ satisfying (1.9)$\}$.
Then $(\mathcal{L}_{p,q}(X, Y^*), L_{p,q})$ is a Banach space, called the space of $(p, q)-$Factorable operators.

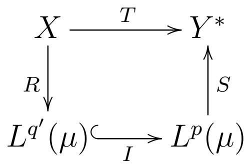
For a general definition of the spaces $\mathcal{L}_{p,q}(X, Y)$ (including the case when the range is not a dual space), see [DF93, Chapter 17].

Since Y^* is 1-complemented in its bidual, [DF93, Theorem 18.11] gives the following result.

Theorem 1.8. *Let $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \geq 1$. Let $T \in \mathcal{B}(X, Y^*)$. The following two statements are equivalent :*

(i) $T \in \mathcal{L}_{p,q}(X, Y^*)$.

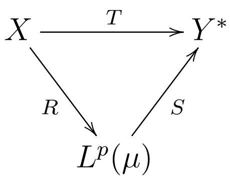
(ii) There are a measure space (Ω, μ) (a probability space when $\frac{1}{p} + \frac{1}{q} > 1$) and operators $R \in \mathcal{B}(X, L^{q'}(\mu))$ and $S \in \mathcal{B}(L^p(\mu), Y^*)$ such that $T = S \circ I \circ R$



where $I : L^{q'}(\mu) \to L^p(\mu)$ *is the inclusion mapping (well defined because* $q' \geq p$*).*

In this case, $L_{p,q}(T) = \inf \|S\|\|R\|$ *over all such factorizations.*

Remark 1.9. *Here we consider the case when* $\frac{1}{p} + \frac{1}{q} = 1$*. Denote by* p' *the conjugate exponent of* p*. We have* $T \in \mathcal{L}_{p,p'}(X, Y^*)$ *if and only if there are a measure space* (Ω, μ)*, operators* $R \in \mathcal{B}(X, L^p(\mu))$ *and* $S \in \mathcal{B}(L^p(\mu), Y^*)$ *such that* $T = SR$



We usually write $\Gamma_p(X, Y^*)$ *instead of* $\mathcal{L}_{p,p'}(X, Y^*)$ *and the norm of an element* $T \in \Gamma_p(X, Y^*)$ *is denoted by* $\gamma_p(T)$*. Such operators are called* $p-$*factorable. It follows from the very definition of* $(p, q)-$*Factorable operators that* $\Gamma_p(X, Y^*)$ *is a dual space whose predual is* $X \otimes_{\alpha_{p,p'}} Y$*.*

If X and Y are finite dimensional, it follows from the very definition of the dual norm that

$$X \otimes_{\alpha'_{p,q}} Y = (X^* \otimes_{\alpha_{p,q}} Y^*)^*.$$

The next theorem describes, for any Banach spaces E and F, the elements of the space $(E \otimes_{\alpha_{p,q}} F)^*$.

Theorem 1.10. *[DF93, Theorem 19.2] Let* E *and* F *be Banach spaces. Let* $p, q \in [1, \infty]$ *with* $\frac{1}{p} + \frac{1}{q} \geq 1$ *and* $K \subset B_{E^*}$ *and* $L \subset B_{F^*}$ *weak*$-*$*-compact norming sets for E and F, respectively. For* $\phi : E \otimes F \to \mathbb{C}$ *the following two statements are equivalent:*

(i) $\phi \in (E \otimes_{\alpha_{p,q}} F)^*$.

(ii) There are a constant $A \geq 0$ *and probability measures* μ *on* K *and* ν *on* L *such that for all* $x \in E$ *and* $y \in F$*,*

$$|\langle \phi, x \otimes y \rangle| \leq A \left(\int_K |\langle x^*, x \rangle|^{q'} d\mu(x^*) \right)^{1/q'} \left(\int_L |\langle y^*, y \rangle|^{p'} d\mu(y^*) \right)^{1/p'} \tag{1.10}$$

(if the exponent is ∞*, we replace the integral by the norm).*

In this case, $\|\phi\|_{(E \otimes_{\alpha_{p,q}} F)^*} = \inf \{A \mid A \text{ as in (ii)}\}$.

This theorem will allow us to describe the predual of $\mathcal{L}_{p,q}(\ell_1^n, \ell_\infty^m)$, $n, m \in \mathbb{N}$. Let us apply the previous theorem with $E = \ell_\infty^n$ and $F = \ell_\infty^m$. Take $T \in \ell_1^n \otimes_{\alpha'_{p,q}} \ell_1^m = (\ell_\infty^n \otimes_{\alpha_{p,q}}$

$\ell_\infty^m)^*$ and let

$$T = \sum_{i=1}^{n}\sum_{j=1}^{m} T(i,j) e_i \otimes e_j$$

be a representation of T. In the previous theorem, we can take $K = \{1, 2, \dots, n\}$ and $L = \{1, 2, \dots, m\}$. In this case, a probability measure μ on K is nothing but a sequence $\mu = (\mu_1, \dots, \mu_n)$ where, for all i, $\mu_i := \mu(\{i\}) \geq 0$ and $\sum_i \mu_i = 1$. Similarly, $\nu = (\nu_1, \dots, \nu_m)$ where, for all i, $\nu_i \geq 0$ and $\sum_i \nu_i = 1$. In this case, the inequality (1.10) means that for all sequences of complex numbers $x = (x_i)_{i=1}^n, y = (y_j)_{i=j}^m$,

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m} T(i,j) x_i y_j\right| \leq A \left(\sum_{k=1}^{n} |x_k|^{q'} \mu_k\right)^{1/q'} \left(\sum_{k=1}^{m} |y_k|^{p'} \nu_k\right)^{1/p'}.$$

Set $\alpha_k = x_k \mu_k^{1/q'}$, $\beta_k = y_k \nu_k^{1/p'}$ and define, for $1 \leq i \leq n, 1 \leq j \leq m$, $c(i,j)$ such that $T(i,j) = c(i,j)\mu_i^{1/q'}\nu_j^{1/p'}$ (we can assume $\mu_i > 0$ and $\nu_j > 0$). Then, the previous inequality becomes

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m} c(i,j) \beta_j \alpha_i\right| \leq A \|\alpha\|_{\ell_{q'}^n} \|\beta\|_{\ell_{p'}^m}.$$

This means that the operator $c : \ell_{q'}^n \to \ell_p^m$ whose matrix is $[c(i,j)]_{1\leq j\leq m, 1\leq i\leq n}$ has a norm smaller than A. Moreover, if we see T as a mapping from ℓ_∞^n into ℓ_1^m the relation between T and c means that T admits the following factorization

$$\begin{array}{ccc} \ell_\infty^n & \xrightarrow{T} & \ell_1^m \\ {\scriptstyle d_\mu}\downarrow & & \uparrow{\scriptstyle d_\nu} \\ \ell_{q'}^n & \xrightarrow[c]{} & \ell_p^m \end{array}$$

where d_μ and d_ν are the operators of multiplication by $\mu = (\mu_1^{1/q'}, \dots, \mu_n^{1/q'})$ and $\nu = (\nu_1^{1/p'}, \dots, \nu_m^{1/p'})$. Those operators have norm 1.

Therefore, it is easy to check that

$$\|T\|_{(\ell_\infty^n \otimes_{\alpha_{p,q}} \ell_\infty^m)^*} = \inf\left\{\|c\| \mid T = d_\nu \circ c \circ d_\mu\right\}. \tag{1.11}$$

The elements of $(\ell_\infty^n \otimes_{\alpha_{p,q}} \ell_\infty^m)^*$ are called $(q', p')-$dominated operators. For more informations about this space in the infinite dimensional case (it is the predual of $\mathcal{L}_{p,q}$), see for instance [DF93, Chapter 19].

By (1.11) and the fact that $\mathcal{L}_{p,q}(\ell_1^n, \ell_\infty^n) = (\ell_1^n \otimes_{\alpha'_{p,q}} \ell_1^m)^*$, we get the following result.

Proposition 1.11. *Let $v = [v_{ij}] : \ell_1^n \to \ell_\infty^m$. Then*

$$L_{p,q}(v) = \sup |Tr(vu)|$$

where the supremum runs over all $u : \ell_\infty^m \to \ell_1^n$ *admitting the factorization*

$$\begin{array}{ccc} \ell_\infty^m & \xrightarrow{u} & \ell_1^n \\ {\scriptstyle d_\mu}\downarrow & & \uparrow{\scriptstyle d_\nu} \\ \ell_{p'}^m & \xrightarrow[c]{} & \ell_q^n \end{array}$$

with $\|d_\mu\| \leq 1$, $\|d_\nu\| \leq 1$ *and* $\|c\| \leq 1$.
Equivalently,

$$L_{p,q}(v) = \sup\left\{ \left| \sum_{i=1}^m \sum_{j=1}^n v_{ij} c_{ji} \mu_i \nu_j \right| \mid \|c : \ell_{p'}^m \to \ell_q^n\| \leq 1, \|\mu\|_{\ell_{p'}^m} \leq 1, \|\nu\|_{\ell_{q'}^n} \leq 1 \right\}.$$

1.2 Schatten classes

1.2.1 Definition and duality

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let tr be the trace on $\mathcal{B}(\mathcal{K})$. We let, for $1 \leq p < +\infty$, $\mathcal{S}^p(\mathcal{K}, \mathcal{H})$ denote the Schatten classes class of order p equipped with the norm $\|.\|_p$ defined for an operator $T\colon \mathcal{K} \to \mathcal{H}$ by

$$\|T\|_p = \mathrm{tr}(|T|^p)^{1/p},$$

where $|T| = (T^*T)^{\frac{1}{2}}$. We will also denote by $\mathcal{S}^\infty(\mathcal{K}, \mathcal{H})$ the space of compact operators from $\mathcal{K}$ into $\mathcal{H}$.

We recall the duality theorem for Schatten classes.

Theorem 1.12. *Let* $1 < p < +\infty$ *and let* q *to be the conjugate exponent of* p. *Then*

$$\begin{array}{ccc} \mathcal{S}^q(\mathcal{H}, \mathcal{K}) & \longrightarrow & \mathcal{S}^p(\mathcal{K}, \mathcal{H})^* \\ T & \longmapsto & \mathrm{tr}(T.) \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{B}(\mathcal{H}, \mathcal{K}) & \longrightarrow & \mathcal{S}^1(\mathcal{K}, \mathcal{H})^* \\ T & \longmapsto & \mathrm{tr}(T.) \end{array}$$

are isometric isomorphisms.

For a proof of this theorem and several properties of Schatten classes, see for instance [Zhu90].

We will mainly work with $\mathcal{S}^1(\mathcal{K}, \mathcal{H})$, $\mathcal{S}^2(\mathcal{K}, \mathcal{H})$ and $\mathcal{S}^\infty(\mathcal{K}, \mathcal{H})$. Note that $\mathcal{S}^1(\mathcal{K}, \mathcal{H})$, the *trace class operators*, is the smallest space among all Schatten classes. This comes from the fact that, for all $1 \leq p_1 \leq p_2 < +\infty$,

$$\|\cdot\|_{p_2} \leq \|\cdot\|_{p_1}.$$

For any h_1, h_2 in $\mathcal{H}$, we may identify $\overline{h_1} \otimes h_2$ with the operator $h \mapsto \langle h, h_1 \rangle h_2$ from $\mathcal{H}$ into $\mathcal{H}$. This yields an identification of $\overline{\mathcal{H}} \otimes \mathcal{H}$ with the space of finite rank operators on $\mathcal{H}$, and this identification extends to an isometric isomorphism

$$\overline{\mathcal{H}} \widehat{\otimes} \mathcal{H} \; = \; S^1(\mathcal{H}), \tag{1.12}$$

see e.g. [Pal01, p. 837].

Using operator space theory and the Haagerup tensor product introduced in Subsection 1.1.3, we have, by [ER00, Proposition 9.3.4], a complete isometry

$$(\mathcal{K}_c)^* \overset{h}{\otimes} H_c = \mathcal{S}^1(\mathcal{K}, \mathcal{H}). \tag{1.13}$$

Similarly, we have a complete isometry

$$\mathcal{H}_c \overset{h}{\otimes} (\mathcal{K}_c)^* = \mathcal{S}^\infty(\mathcal{K}, \mathcal{H}). \tag{1.14}$$

Note that $\mathcal{S}^2(\mathcal{K}, \mathcal{H})$ is a Hilbert space, for the inner product

$$\langle S, T\rangle := \mathrm{tr}(ST^*),$$

and elements of $\mathcal{S}^2(\mathcal{K}, \mathcal{H})$ are called *Hilbert-Schmidt operators*.

Remark 1.13. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be two σ-finite measure spaces. If $J \in L^2(\Omega_1 \times \Omega_2)$, the operator*

$$\begin{array}{rccc} X_J : & L^2(\Omega_1) & \longrightarrow & L^2(\Omega_2) \\ & r & \longmapsto & \displaystyle\int_{\Omega_1} J(t, .)r(t)\, d\mu_1(t) \end{array} \tag{1.15}$$

is a Hilbert-Schmidt operator and $\|X_J\|_2 = \|J\|_2$. Moreover, any element of $\mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2))$ has this form (see e.g. [Woj91]). We summarize these facts by writing an isometric identification

$$L^2(\Omega_1 \times \Omega_2) = \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2)). \tag{1.16}$$

In this thesis, we will often work with the finite dimensional versions of the Hilbert-Schmidt and the trace class operators. For $n \geq 2$, denote by $\mathcal{S}^1_n$ the space of $n \times n$ matrices equipped with the trace norm and by $\mathcal{S}^2_n$ the space of $n \times n$ matrices equipped with the Hilbert-Schmidt norm.

1.2.2 Tensor products of Hilbert space operators and trace duality

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces.

We may consider

$$\mathcal{B}(\mathcal{H}, \mathcal{K}) \subset \mathcal{B}(\mathcal{H} \overset{2}{\oplus} \mathcal{K}) \tag{1.17}$$

by identifying any $S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with the matrix $\begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}$. This is an isometric inclusion. Then for any von Neumann algebra $\mathcal{M}$, we let

$$\mathcal{M}\overline{\otimes}\mathcal{B}(\mathcal{H}, \mathcal{K}) \tag{1.18}$$

be the w^*-closure of $\mathcal{M} \otimes \mathcal{B}(\mathcal{H}, \mathcal{K})$ in the von Neumann algebra $\mathcal{M}\overline{\otimes}\mathcal{B}(\mathcal{H} \overset{2}{\oplus} \mathcal{K})$. Likewise for any two other Hilbert spaces $\mathcal{H}'$ and $\mathcal{K}'$ we let $\mathcal{B}(\mathcal{H}', \mathcal{K}')\overline{\otimes}\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the w^*-closure of $\mathcal{B}(\mathcal{H}', \mathcal{K}') \otimes \mathcal{B}(\mathcal{H}, \mathcal{K})$ in the von Neumann algebra $\mathcal{B}(\mathcal{H}' \overset{2}{\oplus} \mathcal{K}')\overline{\otimes}\mathcal{B}(\mathcal{H} \overset{2}{\oplus} \mathcal{K})$.

Let $\mathcal{H}' \overset{2}{\otimes} \mathcal{H}$ denote the Hilbertian tensor product of $\mathcal{H}'$ and $\mathcal{H}$. As is well-known, the natural embedding $\mathcal{B}(\mathcal{H}',\mathcal{K}') \otimes \mathcal{B}(\mathcal{H},\mathcal{K}) \subset \mathcal{B}(\mathcal{H}' \overset{2}{\otimes} \mathcal{H}, \mathcal{K}' \overset{2}{\otimes} \mathcal{K})$ extends to an isometric identification

$$\mathcal{B}(\mathcal{H}',\mathcal{K}')\overline{\otimes}\mathcal{B}(\mathcal{H},\mathcal{K}) = \mathcal{B}(\mathcal{H}' \overset{2}{\otimes} \mathcal{H}, \mathcal{K}' \overset{2}{\otimes} \mathcal{K}). \tag{1.19}$$

For any $T \in \mathcal{S}^1(\mathcal{K},\mathcal{H})$ and $T' \in \mathcal{S}^1(\mathcal{K}',\mathcal{H}')$, the operator $T' \otimes T$ belongs to the space $\mathcal{S}^1(\mathcal{K}' \overset{2}{\otimes} \mathcal{K}, \mathcal{H}' \overset{2}{\otimes} \mathcal{H})$. This yields an embedding of the tensor product $\mathcal{S}^1(\mathcal{K}',\mathcal{H}') \otimes \mathcal{S}^1(\mathcal{K},\mathcal{H})$ into $\mathcal{S}^1(\mathcal{K}' \overset{2}{\otimes} \mathcal{K}, \mathcal{H}' \overset{2}{\otimes} \mathcal{H})$. Let γ denote the norm on $\mathcal{S}^1(\mathcal{K}',\mathcal{H}') \otimes \mathcal{S}^1(\mathcal{K},\mathcal{H})$ induced by this embedding. Namely for any finite families $(T_j)_j$ in $\mathcal{S}^1(\mathcal{K},\mathcal{H})$ and $(T_j')_j$ in $\mathcal{S}^1(\mathcal{K}',\mathcal{H}')$, $\gamma\big(\sum_j T_j' \otimes T_j\big)$ is the trace norm of the operator $\mathcal{K}' \overset{2}{\otimes} \mathcal{K} \to \mathcal{H}' \overset{2}{\otimes} \mathcal{H}$ taking $\sum_i x_i' \otimes x_i$ to the sum $\sum_{i,j} T_j'(x_i') \otimes T_j(x_i)$ for all finite families $(x_i)_i$ in $\mathcal{K}$ and $(x_i')_i$ in $\mathcal{K}'$. Next we let $\mathcal{S}^1(\mathcal{K}',\mathcal{H}') \overset{\gamma}{\otimes} \mathcal{S}^1(\mathcal{K},\mathcal{H})$ denote the completion of the resulting normed space $\big(\mathcal{S}^1(\mathcal{K}',\mathcal{H}')\otimes\mathcal{S}^1(\mathcal{K},\mathcal{H}),\gamma\big)$. Since finite rank operators are dense in trace class operators, the algebraic tensor product $\mathcal{S}^1(\mathcal{K}',\mathcal{H}')\otimes\mathcal{S}^1(\mathcal{K},\mathcal{H})$ is dense in $\mathcal{S}^1(\mathcal{K}' \overset{2}{\otimes} \mathcal{K}, \mathcal{H}' \overset{2}{\otimes} \mathcal{H})$. Hence we actually have an isometric identification

$$\mathcal{S}^1(\mathcal{K}',\mathcal{H}') \overset{\gamma}{\otimes} \mathcal{S}^1(\mathcal{K},\mathcal{H}) = \mathcal{S}^1(\mathcal{K}' \overset{2}{\otimes} \mathcal{K}, \mathcal{H}' \overset{2}{\otimes} \mathcal{H}).$$

Then trace duality given in Theorem 1.12 yields an identification $\big(\mathcal{S}^1(\mathcal{K}',\mathcal{H}')\overset{\gamma}{\otimes}\mathcal{S}^1(\mathcal{K},\mathcal{H})\big)^* = \mathcal{B}(\mathcal{H}' \overset{2}{\otimes} \mathcal{H}, \mathcal{K}' \overset{2}{\otimes} \mathcal{K})$ and hence, by (1.19), we have

$$\big(\mathcal{S}^1(\mathcal{K}',\mathcal{H}') \overset{\gamma}{\otimes} \mathcal{S}^1(\mathcal{K},\mathcal{H})\big)^* = \mathcal{B}(\mathcal{H}',\mathcal{K}')\overline{\otimes}\mathcal{B}(\mathcal{H},\mathcal{K}). \tag{1.20}$$

For any $\eta \in \mathcal{K}$ and $\xi \in \mathcal{H}$, we let $\overline{\eta}\otimes\xi\colon \mathcal{K} \to \mathcal{H}$ denote the operator taking any $z \in \mathcal{K}$ to $\langle z,\eta\rangle\xi$. Then $\overline{\mathcal{K}} \otimes \mathcal{H}$ identifies with the space of finite rank operators from $\mathcal{K}$ into $\mathcal{H}$. We let $\Phi\colon \mathcal{S}^2(\overline{\mathcal{H}'},\mathcal{H}) \otimes \mathcal{S}^2(\mathcal{K},\overline{\mathcal{K}'}) \to \mathcal{S}^1(\mathcal{K}',\mathcal{H}') \overset{\gamma}{\otimes} \mathcal{S}^1(\mathcal{K},\mathcal{H})$ be the unique linear mapping satisfying

$$\Phi\big((\xi' \otimes \xi) \otimes (\overline{\eta} \otimes \overline{\eta'})\big) = \overline{\eta'} \otimes \xi' \otimes \overline{\eta} \otimes \xi, \qquad \xi \in \mathcal{H},\ \xi' \in \mathcal{H}',\ \eta \in \mathcal{K},\ \eta' \in \mathcal{K}'.$$

Lemma 1.14. *The mapping Φ extends to an isometry (still denoted by)*

$$\Phi\colon \mathcal{S}^2(\overline{\mathcal{H}'},\mathcal{H}) \overset{\wedge}{\otimes} \mathcal{S}^2(\mathcal{K},\overline{\mathcal{K}'}) \longrightarrow \mathcal{S}^1(\mathcal{K}',\mathcal{H}') \overset{\gamma}{\otimes} \mathcal{S}^1(\mathcal{K},\mathcal{H}).$$

We will prove this proposition by approximation. We first need the finite dimensional version of this result.

We let E_{ij} denote the standard matrix units on M_n for $1 \leq i,j \leq n$. We regard M_{n^2} as the space of matrices with columns and rows indexed by $\{1,\ldots,n\}^2$. Thus we write $E_{(i,k),(j,l)}$ for its standard matrix units. Then we let $M_n \otimes_{\min} M_n$ denote the minimal tensor product of two copies of M_n. According to the definition of $\otimes_{\min}$ (see e.g. [Tak79, p. IV.4.8]), the isomorphism $J_0\colon M_n \otimes_{\min} M_n \to M_{n^2}$ given by

$$J_0(E_{ij} \otimes E_{kl}) = E_{(i,k),(j,l)}, \qquad 1 \leq i,j,k,l \leq n, \tag{1.21}$$

is an isometry.

Note that S_n^{1*} is isometrically isomorphic to M_n through the duality pairing

$$S_n^1 \times M_n \to \mathbb{C}, \qquad (A, B) \mapsto \mathrm{Tr}\big({}^t\!AB\big). \tag{1.22}$$

With this convention (note the use of the transpose), the dual basis of $(E_{ij})_{1\leq i,j\leq n}$ is $(E_{ij})_{1\leq i,j\leq n}$ itself. We have

$$\big(S_n^1 \otimes_\gamma S_n^1\big)^* = M_n \otimes_{\min} M_n, \tag{1.23}$$

through the duality pairing (1.22) applied twice.

Lemma 1.15. *The isomorphism* $J\colon S_n^2\widehat{\otimes}S_n^2 \to S_n^1 \otimes_\gamma S_n^1$ *given by*

$$J(E_{ik} \otimes E_{jl}) = E_{ij} \otimes E_{kl}, \qquad 1 \leq i,j,k,l \leq n,$$

is an isometry.

Proof. According to the equality

$$\Big\|\sum_{i,k} c_{ik}E_{ik}\Big\|_2 = \Big(\sum_{i,k} |c_{ik}|^2\Big)^{\frac{1}{2}}, \qquad c_{ik} \in \mathbb{C},$$

we can naturally identify S_n^2 with either $\ell^2_{n^2}$ or its conjugate space. Then applying the identity (1.12) with $\mathcal{H} = \ell^2_{n^2}$, we obtain that the mapping $J_1\colon S_n^2\widehat{\otimes}S_n^2 \to S_{n^2}^1$ given by

$$J_1(E_{ik} \otimes E_{jl}) = E_{(i,k),(j,l)}, \qquad 1 \leq i,j,k,l \leq n,$$

is an isometry.

Now let $J_2\colon S_n^1 \otimes_\gamma S_n^1 \to S_{n^2}^1$ be the isomorphism given by

$$J_2(E_{ij} \otimes E_{kl}) = E_{(i,k),(j,l)}, \qquad 1 \leq i,j,k,l \leq n.$$

Taking into account the identity (1.23), we see that J_2^{-1} is the adjoint of J_0. Consequently, J_2^{-1} is an isometry. Since $J = J_2^{-1}J_1$, we deduce that J is an isometry as well. □

Proof of Proposition 1.14. By approximation, we can assume that the four Hilbert spaces $\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}'$ are finite dimensional, say of dimension $n \geq 1$. In this case, $\mathcal{S}^2(\overline{\mathcal{H}'}, \mathcal{H})$ and $\mathcal{S}^2(\mathcal{K}, \overline{\mathcal{K}'})$ identify with $\mathcal{S}_n^2$, the space of $n \times n$ matrices equipped with the Hilbert-Schmidt norm. Likewise $\mathcal{S}^1(\mathcal{K}, \mathcal{H})$ and $\mathcal{S}^1(\mathcal{K}', \mathcal{H}')$ identify with $\mathcal{S}_n^1$, the space of $n \times n$ matrices equipped with the trace norm. Then under these identifications, $\Phi\colon \mathcal{S}_n^2 \otimes \mathcal{S}_n^2 \to \mathcal{S}_n^1 \otimes \mathcal{S}_n^1$ is given by

$$\Phi(E_{ki} \otimes E_{jl}) = E_{ij} \otimes E_{kl}, \qquad 1 \leq i,j,k,l \leq n.$$

Thefore, since the transposition is an isometry of $\mathcal{S}_n^2$, the result follows from Lemma 1.15. □

1.3 L^p_σ-spaces and duality

Let (Ω, μ) be a σ-finite measure space and let E be a Banach space. For any $1 \leq p \leq +\infty$, we let $L^p(\Omega; E)$ denote the classical Bochner space of measurable functions $\varphi \colon \Omega \to E$ (defined up to almost everywhere zero functions) such that the norm function $\|\varphi(\cdot)\|$ belongs to $L^p(\Omega)$ (see e.g. [DU79, Chapter II]).

We will consider a dual version. Assume that E is separable. A function $\phi \colon \Omega \to E^*$ is said to be w^*-measurable if for all $x \in E$, the function $t \in \Omega \mapsto \langle \phi(t), x\rangle$ is measurable. In this case, the function $t \in \Omega \mapsto \|\phi(t)\|$ is measurable. Indeed, if $(x_n)_n$ is a dense sequence in the unit sphere of E, then $\|\phi(.)\| = \sup_n |\langle \phi(.), x_n\rangle|$ is the supremum of a sequence of measurable functions, hence is measurable.

Let $1 \leq q \leq +\infty$. By definition, $L^q_\sigma(\Omega; E^*)$ is the space of all w^*-measurable $\phi \colon \Omega \to E^*$ such that $\|\phi(.)\| \in L^q(\Omega)$, after taking quotient by the functions which are equal to 0 almost everywhere. We equip this space with

$$\|\phi\|_q = \|\|\phi(.)\|\|_{L^q(\Omega)}.$$

Then $(L^q_\sigma(\Omega; E^*), \|.\|_q)$ is a Banach space and by construction, $L^q(\Omega; E^*) \subset L^q_\sigma(\Omega; E^*)$ isometrically.

Suppose that $1 \leq p < +\infty$ and let $1 < q \leq +\infty$ be the conjugate exponent of p. For any $\phi \in L^q_\sigma(\Omega; E^*)$ and any $\varphi \in L^p(\Omega; E)$, the function $t \mapsto \langle \phi(t), \varphi(t)\rangle$ is integrable, which yields a duality pairing

$$\langle \phi, \varphi\rangle := \int_\Omega \langle \phi(t), \varphi(t)\rangle \, \mathrm{d}\mu(t)\,. \tag{1.24}$$

Moreover by Hölder's inequality, we have

$$|\langle \phi, \varphi\rangle| \leq \|\phi\|_q \|\varphi\|_p. \tag{1.25}$$

Theorem 1.16. *The duality pairing (1.24) induces an isometric isomorphism*

$$L^p(\Omega; E)^* = L^q_\sigma(\Omega; E^*). \tag{1.26}$$

The above theorem is well-known and has extensions to the non separable case. However we havent found a satisfactory reference for this simple (=separable) case and provide a proof below for the sake of completeness. See [DU79, Chapter IV] and the references therein for more information.

Recall that we have $L^1(\Omega; E)^* = \mathcal{B}(L^1(\Omega), E^*)$ by (1.4). Hence in the case $p = 1$, the above theorem yields an isometric identification

$$L^\infty_\sigma(\Omega; F^*) = B(L^1(\Omega), F^*), \tag{1.27}$$

a classical result going back to [DP40, Theorem 2.1.6].

Proof of Theorem 1.16. The inequality (1.25) yields a contractive map $\kappa \colon L^q_\sigma(\Omega; E^*) \to L^p(\Omega; E)^*$. Our aim is to show that κ is an isometric isomorphism.

According to the separability assumption there exists a nondecreasing sequence $(E_n)_{n\geq 1}$ of finite dimensional subspaces of E such that $\cup_n E_n$ is dense in E. Since E_n is finite dimensional, $L^q_\sigma(\Omega, E_n^*) = L^q(\Omega, E_n^*)$ and E_n satisfies the conclusion of the theorem to be proved, that is,

$$L^p(\Omega; E_n)^* = L^q(\Omega; E_n^*) \tag{1.28}$$

isometrically (see [DU79, Chapter IV]). In the sequel we regard $L^p(\Omega; E_n)$ as a subspace of $L^p(\Omega; E)$ in a natural way.

We first note that κ is 1-1. Indeed if $\phi \in L^q_\sigma(\Omega; E^*)$ is such that $\kappa(\phi) = 0$, then for any $n \geq 1$, $\phi(t)_{|E_n} = 0$ a.-e. by (1.28). Hence $\phi(t)_{|\cup_n E_n} = 0$ a.-e., which implies that $\phi(t) = 0$ a.-e..

Now let $\delta \in L^p(\Omega; E)^*$, with $\|\delta\| \leq 1$. Applying (1.28) to the restriction of δ to $L^p(\Omega; E_n)$ we obtain, for any $n \geq 1$, a measurable function $\phi_n \colon \Omega \to E_n^*$ such that $\|\phi_n\|_q \leq 1$ and

$$\forall\, \varphi \in L^p(\Omega) \otimes E_n, \qquad \delta(\varphi) = \int_\Omega \langle \phi_n(t), \varphi(t) \rangle \, \mathrm{d}\mu(t)\,.$$

We may assume that for any $n \geq 1$, we have

$$\forall\, t \in \Omega, \qquad \phi_{n+1}(t)_{|E_n} = \phi_n(t). \tag{1.29}$$

Indeed by construction, $\phi_{n+1|E_n} = \phi_n$ a.-e. and the family $(\phi_n)_{n\geq 1}$ is countable so we can modify all the functions ϕ_n on a common negligible set to get (1.29).

It follows that for any $t \in \Omega$, $(\|\phi_n(t)\|)_{n\geq 1}$ is a nondecreasing sequence, so we can define a measurable $\nu \colon \Omega \to [0, \infty]$ by

$$\nu(t) = \lim_n \|\phi_n(t)\|, \qquad t \in \Omega.$$

If $q < \infty$ we may write

$$\int_\Omega \nu(t)^q \, \mathrm{d}\mu(t) \; = \lim_n \int_\Omega \|\phi_n(t)\|^q \, \mathrm{d}\mu(t) \; \leq 1,$$

by the monotone convergence Theorem. This implies that ν is a.-e. finite. If $q = \infty$, the fact that $\|\phi_n\|_\infty \leq 1$ for any $n \geq 1$ implies that $\nu(t) \leq 1$ for a.-e. $t \in \Omega$. Thus in any case, there exists a negligible subset $\Omega_0 \subset \Omega$ such that $\nu(t) < \infty$ for any $t \in \Omega \setminus \Omega_0$.

If $t \in \Omega \setminus \Omega_0$, then by (1.29) and the density of $\cup_n E_n$, there exists a unique element of E^*, that we call $\phi(t)$, such that

$$\forall\, n \geq 1,\; \forall\, x \in E_n, \qquad \langle \phi(t), x \rangle = \langle \phi_n(t), x \rangle.$$

Next we set $\phi(t) = 0$ for any $t \in \Omega_0$. We thus have a function $\phi \colon \Omega \to E^*$.

Let $x \in E$ and let $(x_j)_j$ be a sequence of $\cup_n E_n$ converging to x. Then $\langle \phi(\cdot\,), x_j \rangle \to \langle \phi(\cdot\,), x \rangle$ pointwise. Moreover for any j, the function $\langle \phi(\cdot\,), x_j \rangle$ is measurable by construction, hence $\langle \phi(\cdot\,), x \rangle$ is measurable. Thus ϕ is w^*-measurable.

Now from the definition of ϕ, we see that δ and $\kappa(\phi)$ coincide on $L^p(\Omega) \otimes E_n$ for any $n \geq 1$. Consequently, $\delta = \kappa(\phi)$. Moreover $\|\phi\|_q = \lim_n \|\phi_n\|_q \leq 1$.

This proves that κ is a metric surjection, and hence an isometric isomorphism. □

Let E and F be two separable Banach spaces. Then their projective tensor product $E \overset{\wedge}{\otimes} F$ is separable. Recall that its dual space is equal to $\mathcal{B}(E, F^*)$. Whenever $\phi \colon \Omega \to$

$\mathcal{B}(E, F^*)$ is a w^*-measurable function, then for any $x \in E$, the function $T_\phi(x)\colon \Omega \to F^*$ defined by

$$\big[T_\phi(x)\big](t) = \big[\phi(t)\big](x), \qquad t \in \Omega, \tag{1.30}$$

is w^*-measurable.

Corollary 1.17. *The mapping $\phi \mapsto T_\phi$ given by (1.30) induces an isometric isomorphism*

$$\mathcal{B}(E, L^\infty_\sigma(\Omega, F^*)) = L^\infty_\sigma(\Omega; \mathcal{B}(E, F^*)).$$

Proof. By Theorem 1.16 for $p = 1$, and by (1.2) and (1.3), we have isometric isomorphisms

$$\begin{aligned}\mathcal{B}(E, L^\infty_\sigma(\Omega; F^*)) &= \big(E \overset{\wedge}{\otimes} L^1(\Omega; F)\big)^* \\ &= \big(E \overset{\wedge}{\otimes} L^1(\Omega) \overset{\wedge}{\otimes} F\big)^* \\ &= L^1(\Omega; E \overset{\wedge}{\otimes} F)^* \\ &= L^\infty_\sigma(\Omega; \mathcal{B}(E, F^*)).\end{aligned}$$

It is easy to check that the correspondence is given by (1.30). □

Remark 1.18. *We already noticed that $L^q_\sigma(\Omega; E^*) = L^q(\Omega; E^*)$ when E is finite dimensional. It turns out that for a general Banach space E, the equality $L^q_\sigma(\Omega; E^*) = L^q(\Omega; E^*)$ is equivalent to E^* having the Radon-Nikodym property, see e.g. [DU79, Chapter IV]. All Hilbert spaces (more generally all reflexive Banach spaces) have the Radon-Nikodym property. Later on we will use this property that for any separable Hilbert space H and any $1 \le q \le \infty$, we have*

$$L^q_\sigma(\Omega; H) = L^q(\Omega; H).$$

Let E be a Banach space with the Radon-Nikodym property. In this case, Remark 1.18 ensures that

$$L^1(\Omega, E)^* = L^\infty(\Omega, E^*).$$

Then equality (1.4) implies that

$$L^\infty(\Omega, E^*) = \mathcal{B}(L^1(\Omega), E^*), \tag{1.31}$$

and the isometric isomorphism is given by

$$\begin{aligned} L^\infty(\Omega, E^*) &\longrightarrow \mathcal{B}(L^1(\Omega), E^*). \\ g &\longmapsto \left[f \in L^1(\Omega) \mapsto \int_\Omega f(t) g(t) \mathrm{d}\mu(t) \right] \end{aligned}$$

Let $\mathcal{H}$ be a separable Hilbert space. It is well-known that the natural embedding of $L^\infty(\Omega) \otimes \mathcal{B}(\mathcal{H})$ into $L^\infty_\sigma(\Omega; \mathcal{B}(\mathcal{H}))$ extends to an isometric and w^*-homeomorphic identification $L^\infty(\Omega)\overline{\otimes}\mathcal{B}(\mathcal{H})) = L^\infty_\sigma(\Omega; \mathcal{B}(\mathcal{H}))$ (see [Sak98, Theorem 1.22.13]). Using definition (1.18), we show that this remains true if $\mathcal{B}(\mathcal{H})$ is replaced by $\mathcal{B}(\mathcal{H}, \mathcal{K})$.

Lemma 1.19. *Let $\mathcal{H}, \mathcal{K}$ be any two separable Hilbert spaces. Then the embedding of $L^\infty(\Omega) \otimes \mathcal{B}(\mathcal{H}, \mathcal{K})$ into $L^\infty_\sigma(\Omega; \mathcal{B}(\mathcal{H}, \mathcal{K}))$ extends to an isometric and w^*-homeomorphic identification*

$$L^\infty(\Omega)\overline{\otimes}\mathcal{B}(\mathcal{H}, \mathcal{K}) = L^\infty_\sigma(\Omega; \mathcal{B}(\mathcal{H}, \mathcal{K})).$$

Proof. Let $\mathbb{H} = \mathcal{H} \overset{2}{\oplus} \mathcal{K}$. We regard $L^\infty(\Omega)\overline{\otimes}\mathcal{B}(\mathcal{H},\mathcal{K})$ and $L^\infty_\sigma(\Omega;\mathcal{B}(\mathcal{H},\mathcal{K}))$ as subspaces of $L^\infty(\Omega)\overline{\otimes}\mathcal{B}(\mathbb{H})$ and $L^\infty_\sigma(\Omega;\mathcal{B}(\mathbb{H}))$, respectively. Further we use the identity $L^\infty(\Omega)\overline{\otimes}\mathcal{B}(\mathbb{H}) = L^\infty_\sigma(\Omega;\mathcal{B}(\mathbb{H}))$ mentioned above. The space $L^\infty_\sigma(\Omega;\mathcal{B}(\mathcal{H},\mathcal{K}))$ is a w^*-closed subspace of the dual space $L^\infty_\sigma(\Omega;\mathcal{B}(\mathbb{H}))$ hence we have $L^\infty(\Omega)\overline{\otimes}\mathcal{B}(\mathcal{H},\mathcal{K}) \subset L^\infty_\sigma(\Omega;\mathcal{B}(\mathcal{H},\mathcal{K}))$.

Conversely, let $\phi \in L^\infty_\sigma(\Omega;\mathcal{B}(\mathcal{H},\mathcal{K}))$. Let $T\colon L^1(\Omega) \to \mathcal{B}(\mathcal{H},\mathcal{K})$ be associated to ϕ by the identification (1.27). Consider a net $(P_\iota)_\iota \subset \mathcal{B}(L^1(\Omega))$ of finite rank contractive projections, converging strongly to the identity map. Write $T_\iota = TP_\iota$ and let $\phi_\iota \in L^\infty_\sigma(\Omega;\mathcal{B}(\mathcal{H},\mathcal{K}))$ be associated to T_ι for any ι. Since T_ι is finite rank, ϕ_ι belongs to $L^\infty(\Omega) \otimes \mathcal{B}(\mathcal{H},\mathcal{K})$. Hence to show that $\phi \in L^\infty(\Omega)\overline{\otimes}\mathcal{B}(\mathcal{H},\mathcal{K})$, it suffices to check that $\phi_\iota \to \phi$ in the w^*-topology of $L^\infty_\sigma(\Omega;\mathcal{B}(\mathcal{H},\mathcal{K}))$. Recall that the latter space is the dual space of $L^1(\Omega) \overset{\wedge}{\otimes} \mathcal{S}^1(\mathcal{K},\mathcal{H})$. For any φ in the algebraic tensor product $L^1(\Omega) \otimes \mathcal{S}^1(\mathcal{K},\mathcal{H})$, we have

$$\langle \phi_\iota, \varphi\rangle \underset{\iota\to\infty}{\longrightarrow} \langle \phi, \varphi\rangle$$

by the definition of ϕ_ι. Since $\|P_\iota\| \leq 1$, we have $\|\phi_\iota\|_\infty \leq \|\phi\|_\infty$ for any ι, hence the above convergence result holds true as well for any $\varphi \in L^1(\Omega) \overset{\wedge}{\otimes} \mathcal{S}^1(\mathcal{K},\mathcal{H})$. □

Remark 1.20. *Let E_1, E_2 be two Banach spaces and let $U\colon E_1^* \to E_2^*$ be a w^*-continuous map. For any $\phi \in L^\infty_\sigma(\Omega;E_1^*)$, the composition map $U\circ\phi\colon \Omega \to E_2^*$ belongs to $L^\infty_\sigma(\Omega;E_2^*)$ and the mapping $\phi \mapsto U\circ\phi$ is a bounded operator from $L^\infty_\sigma(\Omega;E_1^*)$ into $L^\infty_\sigma(\Omega;E_2^*)$, whose norm is equal $\|U\|$. It is easy to check that this mapping is w^*-continuous. If further U is an isometry, then $\phi \mapsto U\circ\phi$ is an isometry as well.*

1.4 Measurable factorization in $L^\infty_\sigma(\Omega;\Gamma_2(E,F^*))$

1.4.1 The main result

The main purpose of this section is to prove Theorem 1.21 below. This result will be applied in Subsection 1.4.2 (and in Subsection 3.3.2) to the study of continuous Schur multipliers (see Chapter 2 for the definition).

We will say that a measure space (Ω,μ) is separable when $L^2(\Omega,\mu)$ is separable. This implies that (Ω,μ) is σ-finite and moreover, $L^p(\Omega,\mu)$ is separable for any $1 \leq p < \infty$.

It follows from Remark 1.9 that for any separable Banach spaces E, F, the space $\Gamma_2(E,F^*)$ is a dual space with a separable predual. If H is a separable Hilbert space, then $\mathcal{B}(E,H)$ and $\mathcal{B}(F,H)$ are also dual spaces with separable predual.

Theorem 1.21. *Let (Ω,μ) be a separable measure space and let E, F be two separable Banach spaces. Let $\phi \in L^\infty_\sigma\big(\Omega;\Gamma_2(E,F^*)\big)$. Then there exist a separable Hilbert space H and two functions*

$$\alpha \in L^\infty_\sigma\big(\Omega;\mathcal{B}(E,H)\big) \qquad \textit{and} \qquad \beta \in L^\infty_\sigma\big(\Omega;\mathcal{B}(F,H)\big)$$

such that $\|\alpha\|_\infty\|\beta\|_\infty \leq \|\phi\|_\infty$ and for any $(x,y) \in E \times F$,

$$\big\langle [\phi(t)](x), y\big\rangle = \big\langle [\alpha(t)](x), [\beta(t)](y)\big\rangle, \qquad \textit{for a.-e. } t \in \Omega. \tag{1.32}$$

We will need two lemmas, in which (Ω,μ) denotes an arbitrary σ-finite measure space.

The first one is a variant of the classical classification of abelian von Neumann algebras. For any $\theta \in L^\infty(\Omega)$, and any Hilbert space H, we let $M_\theta \colon L^2(\Omega; H) \to L^2(\Omega; H)$ denote the multiplication operator taking any $\varphi \in L^2(\Omega; H)$ to $\theta\varphi$.

Lemma 1.22. *Let $\mathcal{H}$ be a separable Hilbert space and let $\pi \colon L^\infty(\Omega) \to B(\mathcal{H})$ be a w^*-continuous $*$-representation. There exist a separable Hilbert space H and an isometric embedding $\rho \colon \mathcal{H} \hookrightarrow L^2(\Omega; H)$ such that for any $\theta \in L^\infty(\Omega)$,*

$$\rho\pi(\theta) = M_\theta \rho.$$

Proof. Since π is w^*-continuous, there exists a measurable subset $\Omega' \subset \Omega$ such that the range of π is isomorphic to $L^\infty(\Omega')$ in the von Neumann algebra sense and π coincides with the restriction map. It therefore follows from [Dav96, Theorem II.3.5] that there exist a measurable partition $\{\Omega_n \,:\, 1 \leq n \leq \infty\}$ of Ω' and a unitary operator

$$\rho_1 \colon \mathcal{H} \longrightarrow \oplus^2_{1\leq n\leq\infty} L^2(\Omega_n; \ell^2_n)$$

such that for any $\theta \in L^\infty(\Omega)$, $\rho_1\pi(\theta)\rho_1^*$ coincides with the multiplication by θ. (Note that in the above decomposition, the index n may be finite or infinite and the notation ℓ^2_∞ stands for ℓ^2.) Let

$$H = \overset{2}{\oplus}_{1\leq n\leq\infty} \ell^2_n$$

and consider the canonical embedding

$$\rho_2 \colon \ \oplus^2_{1\leq n\leq\infty} L^2(\Omega_n; \ell^2_n) \longrightarrow L^2(\Omega; H).$$

Then $\rho = \rho_2\rho_1$ satisfies the lemma. $\square$

It is well-known that for any Hilbert space H, the commutant of

$$L^\infty(\Omega) \simeq L^\infty(\Omega) \otimes I_H \subset \mathcal{B}(L^2(\Omega); H)$$

coincides with $L^\infty(\Omega)\overline{\otimes}B(H)$. The next statement is a generalization of this result to the case when H is replaced by Banach spaces.

We consider two separable Banach spaces W_1, W_2. Note that $\mathcal{B}(W_1, W_2^*)$ is a dual space with separable predual. We say that a linear map

$$T \colon L^2(\Omega; W_1) \longrightarrow L^2_\sigma(\Omega; W_2^*)$$

is a module map provided that

$$\forall\, \varphi \in L^2(\Omega; W_1),\ \forall\, \theta \in L^\infty(\Omega), \qquad T(\theta\varphi) = \theta T(\varphi).$$

Next we generalize the notion of multiplication by an L^∞-function as follows. For any $\Delta \in L^\infty_\sigma\big(\Omega; \mathcal{B}(W_1, W_2^*)\big)$, we define a multiplication operator

$$M_\Delta \colon L^2(\Omega; W_1) \longrightarrow L^2_\sigma(\Omega; W_2^*) \tag{1.33}$$

by setting

$$\big[M_\Delta(\varphi)\big](t) = [\Delta(t)](\varphi(t)), \qquad t \in \Omega,$$

for any $\varphi \in L^2(\Omega;W_1)$. Indeed it is easy to check that the function in the right-hand side of the above equality belongs to $L^2_\sigma(\Omega;W_2^*)$. Moreover

$$\|M_\Delta\| = \|\Delta\|_\infty.$$

Each multiplication operator M_Δ is a module map, as we have

$$M_\Delta(\theta\varphi) = M_{\Delta\theta}(\varphi) = \theta M_\Delta(\varphi)$$

for any $\theta \in L^\infty(\Omega)$. The following lemma is a converse.

Lemma 1.23. *Let $T\colon L^2(\Omega;W_1) \to L^2_\sigma(\Omega;W_2^*)$ be a module map. Then there exists a function $\Delta \in L^\infty_\sigma\big(\Omega;\mathcal{B}(W_1,W_2^*)\big)$ such that $T = M_\Delta$.*

Proof. In the scalar case ($W_1 = W_2 = \mathbb{C}$) this is an elementary result; the proof consists in reducing to this scalar case.

We define a bilinear map $\widehat{T}\colon W_1 \times W_2 \to \mathcal{B}(L^2(\Omega))$ by the following formula. For any $w_1 \in W_1$, $w_2 \in W_2$ and $x \in L^2(\Omega)$, we set

$$\big[\widehat{T}(w_1,w_2)\big](x) = \big\{t \mapsto \big\langle \big[T(x\otimes w_1)\big](t), w_2\big\rangle\big\}.$$

Recall the identification $L^2_\sigma(\Omega;W_2^*) = L^2(\Omega;W_2)^*$ from Theorem 1.16. If we consider T as a map from $L^2(\Omega;W_1)$ into $L^2(\Omega;W_2)^*$, then we have

$$\big\langle T(x\otimes w_1), y\otimes w_2\big\rangle = \int_\Omega \Big(\big[\widehat{T}(w_1,w_2)\big](x)\Big)(t)\, y(t)\,\mathrm{d}\mu(t) \tag{1.34}$$

for any $w_1 \in W_1$, $w_2 \in W_2$, $x \in L^2(\Omega)$ and $y \in L^2(\Omega)$.

Further for any $\theta \in L^\infty(\Omega)$ and $x \in L^2(\Omega)$, we have

$$\begin{aligned}\big[\widehat{T}(w_1,w_2)\big](\theta x) &= \big\langle \big[T(\theta(x\otimes w_1))\big](\,\cdot\,), w_2\big\rangle\\ &= \big\langle \theta(\cdot)\,\big[T(x\otimes w_1)\big](\,\cdot\,), w_2\big\rangle\\ &= \theta\big[\widehat{T}(w_1,w_2)\big](x),\end{aligned}$$

because T is a module map. Hence $\widehat{T}(w_1,w_2)$ is a module map.

Let us identify $L^\infty(\Omega)$ with the von Neumann subalgebra of $\mathcal{B}(L^2(\Omega))$ consisting of multiplication operators. The above property shows that $\widehat{T}(w_1,w_2)$ is such a multiplication operator for any $w_1 \in Z_1$ and $w_2 \in Z_2$. Hence we may actually regard $\widehat{T}$ as a bilinear map

$$\widehat{T}\colon W_1 \times W_2 \longrightarrow L^\infty(\Omega).$$

Now observe that applying (1.1), (1.2) and (1.27), we have isometric identifications

$$\begin{aligned}\mathcal{B}_2(W_1\times W_2, L^\infty(\Omega)) &= \mathcal{B}(W_1 \overset{\wedge}{\otimes} W_2, L^\infty(\Omega))\\ &= \mathcal{B}(L^1(\Omega), (W_1 \overset{\wedge}{\otimes} W_2)^*)\\ &= \mathcal{B}(L^1(\Omega), \mathcal{B}(W_1,W_2^*))\\ &= L^\infty_\sigma\big(\Omega;\mathcal{B}(W_1,W_2^*)\big).\end{aligned}$$

Let $\Delta \in L^\infty_\sigma\big(\Omega; \mathcal{B}(W_1, W_2^*)\big)$ be corresponding to $\widehat{T}$ in this identification. Then we have

$$\big\langle [\Delta(t)](w_1), w_2\big\rangle = \big(\widehat{T}(w_1, w_2)\big)(t), \qquad w_1 \in W_1,\ w_2 \in W_2,\ t \in \Omega.$$

Thus applying (3.1) we obtain that

$$\begin{aligned}\big\langle T(x \otimes w_1), y \otimes w_2\big\rangle &= \int_\Omega \big\langle [\Delta(t)](w_1), w_2\big\rangle\, x(t)y(t)\, \mathrm{d}\mu(t)\\ &= \big\langle M_\Delta(x \otimes w_1), y \otimes w_2\big\rangle\end{aligned}$$

for any $w_1 \in W_1$, $w_2 \in W_2$, $x \in L^2(\Omega)$ and $y \in L^2(\Omega)$. By the density of $L^2(\Omega) \otimes W_1$ and $L^2(\Omega) \otimes W_2$ in $L^2(\Omega; W_1)$ and $L^2(\Omega; W_2)$, respectively, this implies that $T = M_\Delta$. □

Proof of Theorem 1.21. This proof should be regarded as a 'module version' of the proof of [Pis96, Theorem 3.4]. As in this book we adopt the following notation. For any finite families $(y_j)_j$ and $(x_i)_i$ in E, we write

$$(y_j)_j < (x_i)_i$$

provided that

$$\forall\, \eta \in E^*, \qquad \sum_j |\eta(y_j)|^2 \leq \sum_i |\eta(x_i)|^2.$$

In the sequel we simply write L^2 (resp. L^∞) instead of $L^2(\Omega)$ (resp. $L^\infty(\Omega)$) as there is no risk of confusion. Then we set

$$V = L^2 \otimes E \subset L^2(\Omega; E).$$

We fix some $\phi \in L^\infty_\sigma\big(\Omega; \Gamma_2(E, F^*)\big)$ and we let $C = \|\phi\|_\infty$. Then ϕ is an element of $L^\infty_\sigma\big(\Omega; \mathcal{B}(E, F^*)\big)$. Hence according to (1.33) we may consider the multiplication operator

$$T = M_\phi \colon L^2(\Omega; E) \longrightarrow L^2_\sigma(\Omega; F^*).$$

We let $I = L^\infty \times E^*$. A generic element of I will be denoted by $\zeta = (\theta, \eta)$, with $\theta \in L^\infty$ and $\eta \in E^*$.

For any $v = \sum_s x_s \otimes e_s \in V$ (finite sum) and $\zeta = (\theta, \eta) \in I$, we set

$$\zeta\cdot v = \sum_s \eta(e_s)\theta x_s \ \in\ L^2.$$

Lemma 1.24. *Let $(w_j)_j$ and $(v_i)_i$ be finite families in V such that*

$$\forall\, \zeta \in I, \qquad \sum_j \|\zeta\cdot w_j\|_2^2 \leq \sum_i \|\zeta\cdot v_i\|_2^2. \tag{1.35}$$

Then

$$\sum_j \|T(w_j)\|_2^2 \leq C^2 \sum_i \|v_i\|_2^2. \tag{1.36}$$

Proof. Let $(w_j)_j$ and $(v_i)_i$ be finite families in V and assume (1.35). Consider $e_{i,s}, f_{j,s}$ in E, $x_{i,s}, y_{j,s}$ in L^2 such that

$$v_i = \sum_s x_{i,s}\otimes e_{i,s} \qquad \text{and} \qquad w_j = \sum_s y_{j,s}\otimes f_{j,s}.$$

Let $\zeta=(\theta,\eta)\in I$. For any j,

$$\|\zeta\cdot w_j\|_2^2 \;=\; \int_\Omega \Bigl|\sum_s \eta(f_{j,s})\theta(t)y_{j,s}(t)\Bigr|^2\, d\mu(t)\,.$$

Hence

$$\sum_j \|\zeta\cdot w_j\|_2^2 \;=\; \int_\Omega |\theta(t)|^2\Bigl(\sum_j\Bigl|\sum_s \eta(f_{j,s})y_{j,s}(t)\Bigr|^2\Bigr)\, d\mu(t)\,.$$

Likewise,

$$\sum_i \|\zeta\cdot v_i\|_2^2 \;=\; \int_\Omega |\theta(t)|^2\Bigl(\sum_i\Bigl|\sum_s \eta(x_{i,s})e_{i,s}(t)\Bigr|^2\Bigr)\, d\mu(t)\,.$$

Thus by (1.35), we have

$$\int_\Omega |\theta(t)|^2\Bigl(\sum_j\bigl|\eta\bigl(w_j(t)\bigr)\bigr|^2\Bigr)\, d\mu(t) \;\le\; \int_\Omega |\theta(t)|^2\Bigl(\sum_i\bigl|\eta\bigl(v_i(t)\bigr)\bigr|^2\Bigr)\, d\mu(t)\,. \tag{1.37}$$

Let $E_1\subset E$ be the subspace spanned by the $e_{i,s}$ and $f_{j,s}$. Since it is finite dimensional, its dual space is obviously separable. Let $(\eta_n)_{n\geq 1}$ be a dense sequence of E_1^* and for any $n\geq 1$, extend η_n to an element of E^* (still denoted by η_n). Then for any finite families $(f_j)_j$ and $(e_i)_i$ in E_1, we have

$$(y_j)_j < (x_i)_i \iff \forall\, n\geq 1,\quad \sum_j |\eta_n(f_j)|^2 \leq \sum_i |\eta_n(e_i)|^2.$$

It follows from (1.37) that for almost every $t\in\Omega$, we have

$$\sum_j |\eta_n\bigl(w_j(t)\bigr)|^2 \leq \sum_i |\eta_n\bigl(v_i(t)\bigr)|^2$$

for every $n\geq 1$. Since the functions v_i, w_j are valued in E_1, this implies that

$$(w_j(t))_j < (v_i(t))_i \qquad \text{for a.e. } t\in\Omega.$$

By the implication '(i) ⇒ (iii)' of [Pis96, Theorem 3.4], this property implies that for a.e. $t\in\Omega$,

$$\sum_j \bigl\|[\phi(t)]\bigl(w_j(t)\bigr)\bigr\|_{F^*}^2 \leq C^2\sum_i \bigl\|v_i(t)\bigr\|_E^2.$$

Integrating this inequality on Ω yields (1.36). □

We let Λ be the set of all functions $g\colon I \to \mathbb{R}$ for which there exists a finite family $(v_i)_i$ in V such that

$$\forall\, \zeta \in I, \qquad |g(\zeta)| \leq \sum_i \|\zeta\cdot v_i\|_2^2. \tag{1.38}$$

This is a real vector space. We let Λ_+ denote its positive part, i.e. the set of all functions $I \to \mathbb{R}_+$ belonging to Λ. This is a convex cone. For any $g \in \Lambda$ we set

$$p(g) = C^2 \inf\Big\{\sum_i \|v_i\|_2^2\Big\},$$

where the infimum runs over all finite families $(v_i)_i$ in V satisfying (1.38). Next for any $g \in \Lambda_+$, we set

$$q(g) = \sup\Big\{\sum_j \|T(w_j)\|_2^2\Big\},$$

where the supremum runs over all finite families $(w_j)_j$ in V satisfying

$$\forall \zeta \in I, \qquad g(\zeta) \geq \sum_j \|\zeta\cdot w_j\|_2^2. \tag{1.39}$$

It is easy to check that p is sublinear on Λ and that q is superlinear on Λ_+. Further by Lemma 1.24, $q \leq p$ on Λ_+. Hence by the Hahn-Banach Theorem given in [Pis96, Corollary 3.2], there exists a positive linear functional $\ell\colon \Lambda \to \mathbb{R}$ such that

$$\forall\, g \in \Lambda, \qquad \ell(g) \leq p(g) \tag{1.40}$$

and

$$\forall\, g \in \Lambda_+, \qquad q(g) \leq \ell(g). \tag{1.41}$$

Following [Pis96, Chapter 8], we introduce a Hilbert space

$$\Lambda_2(I, \ell; L^2)$$

defined as follows. First we let $\mathcal{L}(I, \ell; L^2)$ be the set of all functions $G\colon I \to L^2$ such that the $\mathbb{R}$-valued function $\zeta \mapsto \|G(\zeta)\|_2^2$ belongs to Λ and we set $N(G) = \big(\ell(\zeta \mapsto \|G(\zeta)\|_2^2)\big)^{\frac{1}{2}}$ for any such function. Then $\mathcal{L}(I, \ell; L^2)$ is a complex vector space and N is a Hilbertian seminorm on $\mathcal{L}(I, \ell; L^2)$. Hence the quotient of $\mathcal{L}(I, \ell; L^2)$ by the kernel of N is a pre-Hilbert space. By definition, $\Lambda_2(I, \ell; L^2)$ is the completion of this quotient space.

For any $v \in V$, the function $\zeta \mapsto \zeta\cdot v$ belongs to $\mathcal{L}(I, \ell; L^2)$. Then we define a linear map

$$T_1\colon V \longrightarrow \Lambda_2(I, \ell; L^2)$$

as follows: for any $v \in V$, $T_1(v)$ is the class of $\zeta \mapsto \zeta\cdot v$ modulo the kernel of N. Then we have

$$\begin{aligned}\|T_1(v)\|_L^2 &= \ell\big(\zeta \mapsto \|\zeta\cdot v\|^2\big)\\ &\leq p\big(\zeta \mapsto \|\zeta\cdot v\|_2^2\big)\\ &\leq C^2\|v\|_2^2\end{aligned}$$

by (1.40) and the definition of p. Hence T_1 uniquely extends to a bounded operator

$$T_1\colon L^2(\Omega;E)\longrightarrow \Lambda_2(I,\ell;L^2),\qquad \text{with } \|T_1\|\le C.$$

For any $v\in V$, we have

$$\|T(v)\|_2^2\le q\big(\zeta\mapsto\|\zeta\cdot v\|^2\big)\le \ell\big(\zeta\mapsto\|\zeta\cdot v\|^2\big)=\|T_1(v)\|^2.$$

The resulting inequality $\|T(v)\|_2\le\|T_1(v)\|$ implies the existence of a (necessarily unique) bounded linear operator

$$T_2\colon \overline{T_1(V)}\longrightarrow L^2_\sigma(\Omega;F^*),\qquad \text{with } \|T_2\|\le 1,$$

such that

$$\forall\, v\in V,\qquad T(v)=T_2\big(T_1(v)\big). \tag{1.42}$$

For any $v\in V$ and any $\theta\in L^\infty$, we have

$$\|T_1(\theta v)\|\le\|\theta\|_\infty\|T_1(v)\|. \tag{1.43}$$

Indeed write $v=\sum_s x_s\otimes e_s$, with $e_s\in E$ and $x_s\in L^2$. For any $\gamma\in L^\infty$ and $\eta\in E^*$, we have

$$\Big\|\sum_s\eta(e_s)\gamma\theta x_s\Big\|_2\le\|\theta\|_\infty\Big\|\sum_s\eta(e_s)\gamma x_s\Big\|_2.$$

Hence $\|\zeta\cdot(\theta v)\|\le\|\theta\|_\infty\|\zeta\cdot v\|$ for any $\zeta=(\gamma,\eta)\in I$. Since the functional ℓ is positive on Λ, this implies that $\ell\big(\zeta\mapsto\|\zeta\cdot(\theta v)\|^2\big)\le\|\theta\|_\infty^2\ell\big(\zeta\mapsto\|\zeta\cdot v\|^2\big)$, which yields (1.43).

This inequality implies the existence of a (necessarily unique) linear contraction

$$\pi\colon L^\infty\longrightarrow\mathcal{B}\big(\overline{T_1(V)}\big),$$

such that

$$T_1(\theta v)=\pi(\theta)T_1(v),\qquad v\in L^2(\Omega;E),\ \theta\in L^\infty. \tag{1.44}$$

It is clear that π is a homomorphism, hence a $*$-representation.

Let $\theta\in L^\infty$ and assume that $(\theta_\iota)_\iota$ is a bounded net of L^∞ converging to θ in the w^*-topology. For any $x\in L^2$, $\theta_\iota x\to\theta x$ in L^2 (this uses the boundedness of the net). By the continuity of T_1 this implies that for any $e\in E$, $T_1(\theta_\iota x\otimes e)\to T_1(\theta x\otimes e)$ in $\overline{T_1(V)}$. By linearity, this implies that for any $v\in V$, $T_1(\theta_\iota v)\to T_1(\theta v)$ in $\overline{T_1(V)}$. In other words, $\pi(\theta_\iota)(h)\to\pi(\theta)(h)$ for any $h\in T_1(V)$. Since the net $(\pi(\theta_\iota))_\iota$ is bounded, this implies that $\pi(\theta_\iota)\to\pi(\theta)$ strongly. Hence π is a w^*-continuous $*$-representation.

Recall that E and L^2 are assumed separable, hence the Hilbert space $\overline{T_1(V)}$ is separable. By Lemma 1.22, there exists a separable Hilbert space H and an isometric embedding $\rho\colon\overline{T_1(V)}\hookrightarrow L^2(\Omega;H)$ such that $\rho\pi(\theta)=M_\theta\rho$ for any $\theta\in L^\infty$. Then for any such θ and any $v\in L^2(\Omega;E)$, we have

$$\rho T_1(\theta v)=\big[\rho\pi(\theta)T_1\big](v)=\theta\rho(T_1(v)),$$

by (1.44). This shows that the composed map

$$S_1=\rho T_1\colon L^2(\Omega;E)\longrightarrow L^2(\Omega;H)\quad\text{is a module map.}$$

Define

$$S_2 = T_2\rho^* \colon L^2(\Omega; H) \longrightarrow L^2_\sigma(\Omega; F^*).$$

Let $\theta \in L^\infty(\Omega)$. For any $v \in V$, we have

$$\big[T_2\pi(\theta)\big](T_1(v)) = T_2T_1(\theta v) = T(\theta v) = \theta T(v) = \theta T_2(T_1(v))$$

by (1.44), (1.42) and the fact that T is a module map. This shows that

$$T_2\pi(\theta) = M_\theta T_2.$$

Further we have $\rho^* M_\theta = \big(M_{\overline{\theta}}\rho\big)^* = \big(\rho\pi(\overline{\theta})\big)^* = \pi(\theta)\rho^*$. Hence $M_\theta S_2 = S_2 M_\theta$, that is,

$$S_2 \quad \text{is a module map.}$$

Since $\rho^*\rho$ is equal to the identity of $\overline{T_1(V)}$, it follows from (1.42) that

$$T = S_2S_1.$$

Thus we have constructed a 'module Hilbert space factorization' of T, and this is the main point.

To conclude, let $S_{2*} \colon L^2(\Omega; F) \to L^2(\Omega; H^*)$ be the restriction of the adjoint of S_2 to $L^2(\Omega; F)$. Then S_{2*} is a module map. Now apply Lemma 1.23 to S_1 and S_{2*}. Let $\alpha \in L^\infty_\sigma(\Omega; B(E, H))$ and $\beta \in L^\infty_\sigma(\Omega; B(F, H^*))$ such that S_1 is equal to the multiplication by α and S_{2*} is equal to the multiplication by β. Given any $e \in E$ and $f \in F$, we have

$$\begin{aligned}
\int_\Omega \big\langle [\phi(t)](e), f\big\rangle\, x(t)y(t)\,\mathrm{d}\mu(t) &= \big\langle T(x\otimes e), y\otimes f\big\rangle \\
&= \big\langle S_1(x\otimes e), S_{2*}(y\otimes f)\big\rangle \\
&= \int_\Omega \big\langle [\alpha(t)](e)\, x(t), [\beta(t)](f)\, y(t)\big\rangle\,\mathrm{d}\mu(t) \\
&= \int_\Omega \big\langle [\alpha(t)](e), [\beta(t)](f)\big\rangle\, x(t)y(t)\,\mathrm{d}\mu(t)
\end{aligned}$$

for any $x, y \in L^2$. Applying identification between H^* and H, this proves (1.32). By construction, $\|\alpha\|_\infty \leq C$ and $\|\beta\|_\infty \leq 1$. □

1.4.2 A special case

Let (Ω_1, μ_1),(Ω_2, μ_2) and (Ω_3, μ_3) be three separable measure spaces. We are going to apply Theorem 1.21 with $(\Omega, \mu) = (\Omega_2, \mu_2)$, $E = L^1(\Omega_1)$ and $F = L^1(\Omega_3)$.

To any $\phi \in L^\infty(\Omega_1) \times \Omega_2 \times \Omega_3)$, one may associate $\widetilde{\phi} \in L^\infty_\sigma\big(\Omega_2; \mathcal{B}(L^1(\Omega_1, L^\infty(\Omega_3))\big)$ as follows. For any $r \in L^1(\Omega_1)$,

$$\big[\widetilde{\phi}(t_2)\big](r) = \int_{\Omega_1} \phi(t_1, t_2, \cdot\,)\, r(t_1)\,\mathrm{d}\mu_1(t_1), \qquad t_2 \in \Omega_2. \tag{1.45}$$

According to the obvious identification

$$L^\infty(\Omega_1\times\Omega_2\times\Omega_3)=L^\infty_\sigma\big(\Omega_2;L^\infty(\Omega_1\times\Omega_3)\big)$$

and (1.5), the mapping $\phi\mapsto\widetilde{\phi}$ induces a w^*-homeomorphic isometric identification

$$L^\infty(\Omega_1\times\Omega_2\times\Omega_3)=L^\infty_\sigma\big(\Omega_2;\mathcal{B}(L^1(\Omega_1),L^\infty(\Omega_3))\big),$$

By Remark 1.20, the w^*-continuous contractive embedding of $\Gamma_2(L^1(\Omega_1),L^\infty(\Omega_3))$ into the space $\mathcal{B}(L^1(\Omega_1),L^\infty(\Omega_3))$ induces a w^*-continuous contractive embedding

$$L^\infty_\sigma\big(\Omega_2;\Gamma_2(L^1(\Omega_1),L^\infty(\Omega_3))\big)\subset L^\infty_\sigma\big(\Omega_2;\mathcal{B}(L^1(\Omega_1),L^\infty(\Omega_3))\big).$$

Combining with the preceding identification we obtain a further w^*-continuous contractive embedding

$$L^\infty_\sigma\big(\Omega_2;\Gamma_2(L^1(\Omega_1),L^\infty(\Omega_3))\big)\subset L^\infty(\Omega_1\times\Omega_2\times\Omega_3). \tag{1.46}$$

According to this, we will write $\phi\in L^\infty_\sigma\big(\Omega_2;\Gamma_2(L^1(\Omega_1),L^\infty(\Omega_3))\big)$ when $\widetilde{\phi}$ actually belongs to that space. In this case, for the sake of clarity, we let $\|\phi\|_{\infty,\Gamma_2}$ denote its norm as an element of the latter space. It is greater than or equal to its norm as an element of $L^\infty(\Omega_1\times\Omega_2\times\Omega_3)$.

Theorem 1.25. *Let $\phi\in L^\infty(\Omega_1\times\Omega_2\times\Omega_3)$ and $C\geq 0$.*
Then $\phi\in L^\infty_\sigma\big(\Omega_2;\Gamma_2(L^1(\Omega_1),L^\infty(\Omega_3))\big)$ and $\|\phi\|_{\infty,\Gamma_2}\leq C$ if and only if there exist a Hilbert space H and two functions

$$a\in L^\infty\big(\Omega_1\times\Omega_2;H\big)\qquad\textit{and}\qquad b\in L^\infty\big(\Omega_2\times\Omega_3;H\big)$$

such that $\|a\|_\infty\|b\|_\infty\leq C$ and

$$\phi(t_1,t_2,t_3)=\big\langle a(t_1,t_2),b(t_2,t_3)\big\rangle\qquad\textit{for a.-e. } (t_1,t_2,t_3)\in\Omega_1\times\Omega_2\times\Omega_3. \tag{1.47}$$

Proof. Assume that ϕ belongs to $L^\infty_\sigma\big(\Omega_2;\Gamma_2(L^1(\Omega_1),L^\infty(\Omega_3))\big)$, with $\|\phi\|_{\infty,\Gamma_2}\leq C$. According to Theorem 1.21, there exist a Hilbert space H and two functions

$$\alpha\in L^\infty_\sigma\big(\Omega_2;B(L^1(\Omega_1),H)\big)\qquad\text{and}\qquad\beta\in L^\infty_\sigma\big(\Omega_2;B(L^1(\Omega_3),H)\big)$$

such that for any $r_1\in L^1(\Omega_1)$ and $r_3\in L^1(\Omega_3)$,

$$\big\langle[\widetilde{\phi}(t_2)](r_1),r_3\big\rangle\ =\ \big\langle[\alpha(t_2)](r_1),[\beta(t_2)](r_3)\big\rangle\qquad\text{for a.-e. } t_2\in\Omega_2. \tag{1.48}$$

By (1.3), (1.4) and (1.27) we have isometric identifications

$$\begin{aligned}L^\infty_\sigma\big(\Omega_2;\mathcal{B}(L^1(\Omega_1),H)\big)\ &=L^\infty_\sigma\big(\Omega_2;(L^1(\Omega_1)\overset{\wedge}{\otimes}H^*)^*\big)\\&=\big(L^1(\Omega_2)\overset{\wedge}{\otimes}L^1(\Omega_1)\overset{\wedge}{\otimes}H^*\big)^*\\&=L^1(\Omega_1\times\Omega_2;H^*)^*\\&=L^\infty_\sigma(\Omega_1\times\Omega_2;H).\end{aligned}$$

Morover $L^\infty_\sigma(\Omega_1 \times \Omega_2; H) = L^\infty(\Omega_1 \times \Omega_2; H)$, see Remark 1.18. Hence we finally have an isometric identification

$$L^\infty_\sigma\big(\Omega_2; \mathcal{B}(L^1(\Omega_1), H)\big) \;=\; L^\infty(\Omega_1 \times \Omega_2; H).$$

Likewise we have an isometric identification

$$L^\infty_\sigma\big(\Omega_2; \mathcal{B}(L^1(\Omega_3), H)\big) \;=\; L^\infty(\Omega_2 \times \Omega_3; H).$$

Let $a \in L^\infty(\Omega_1 \times \Omega_2; H)$ and $b \in L^\infty(\Omega_2 \times \Omega_3; H)$ be corresponding to α and β respectively in the above identifications. Then for any $r_1 \in L^1(\Omega_1)$,

$$[\alpha(t_2)](r_1) = \int_{\Omega_1} a(t_1, t_2)\, r_1(t_1)\, \mathrm{d}\mu_1(t_1) \qquad \text{for a.-e. } t_2 \in \Omega_2.$$

Likewise, for any $r_3 \in L^1(\Omega_3)$,

$$[\beta(t_2)](r_3) = \int_{\Omega_3} b(t_2, t_3)\, r_3(t_3)\, \mathrm{d}\mu_3(t_3) \qquad \text{for a.-e. } t_2 \in \Omega_2.$$

Combining (1.48) and (1.45) we deduce that for any $r_1 \in L^1(\Omega_1)$ and $r_3 \in L^1(\Omega_3)$, we have

$$\begin{aligned}\int_{\Omega_1\times\Omega_3} &\langle a(t_1,t_2), b(t_2,t_3)\rangle\, r_1(t_1)\, r_3(t_3)\, \mathrm{d}\mu_1(t_1)\mathrm{d}\mu_3(t_3)\\ &= \big\langle \big[\widetilde{\phi}(t_2)\big](r_1), r_3\big\rangle\\ &= \int_{\Omega_1\times\Omega_3} \phi(t_1,t_2,t_3) r_1(t_1)\, r_3(t_3)\, \mathrm{d}\mu_1(t_1)\mathrm{d}\mu_3(t_3)\end{aligned}$$

for a.-e. $t_2 \in \Omega_2$. This implies (1.47) and shows the 'only if' part.

Assume conversely that (1.47) holds true for some a in $L^\infty(\Omega_1\times\Omega_2; H)$ and some b in $L^\infty(\Omega_1\times\Omega_2; H)$. Using the above identifications, we consider $\alpha \in L^\infty_\sigma\big(\Omega_2; \mathcal{B}(L^1(\Omega_1), H)\big)$ and $\beta \in L^\infty_\sigma\big(\Omega_2; \mathcal{B}(L^1(\Omega_3), H)\big)$ be corresponding to a and b, respectively. Then the above computations lead to (1.48). This identity means that for a.-e. $t_2 \in \Omega_2$, we have a Hilbert space factorisation $\widetilde{\phi}(t_2) = \beta(t_2)^*\alpha(t_2)$. This shows that ϕ belongs to $L^\infty_\sigma\big(\Omega_2; \Gamma_2(L^1(\Omega_1), L^\infty(\Omega_3))\big)$, with $\|\phi\|_{\infty,\Gamma_2} \leq \|a\|_\infty \|b\|_\infty$. □

Chapter 2

Linear Schur multipliers

In this chapter, we are interested in generalizations of well-known results about Schur multipliers. Namely, we extend the definition of classical Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$ and define the continuous Schur multipliers on $\mathcal{B}(L^p, L^q)$. In the case $p = q = 2$, there is a famous characterization of Schur multipliers on $\mathcal{B}(\ell_2)$. A similar characterization also holds in the case $\mathcal{B}(\ell_p)$ (see e.g. [Pis96, Chapter 5]) and in the case of continuous Schur multipliers on $\mathcal{B}(L^2)$ (see e.g. [Spr04]). We recall these facts in the first two sections of this chapter. In the third section, we define Schur multipliers on $\mathcal{B}(L^p, L^q)$ and generalize the characterization of Schur multipliers to this continuous case, using the theory of $(p,q)-$factorable operators introduced in Chapter 1. Note that those results are new, even in the setting of classical Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$. In a fourth section, we will apply the results of Section 2.3 to obtain new inclusion relationships between the spaces of Schur multipliers, extending the work of Bennett in [Ben77].

2.1 Classical Schur multipliers

In this section, we regard elements of $B(\ell_p, \ell_q)$ as infinite matrices in the usual way.

Let $m = (m_{ij})_{i,j\geq 1}$ be a bounded family of complex numbers and let $1 \leq p, q \leq +\infty$. We say that m is a *Schur multiplier on* $\mathcal{B}(\ell_p, \ell_q)$ if for any matrix $[a_{ij}]_{i,j\geq 1}$ in $\mathcal{B}(\ell_p, \ell_q)$, the matrix $[m_{ij}a_{ij}]_{i,j\geq 1}$ defines an element of $\mathcal{B}(\ell_p, \ell_q)$. An application of the Closed Graph theorem shows that m is a Schur multiplier if and only if the mapping

$$\begin{array}{rccc} T_m : & \mathcal{B}(\ell_p, \ell_q) & \longrightarrow & \mathcal{B}(\ell_p, \ell_q) \\ & [a_{ij}]_{i,j\geq 1} & \longmapsto & [m_{ij}a_{ij}]_{i,j\geq 1} \end{array} \tag{2.1}$$

is bounded. By definition, the norm of the Schur multiplier m is the norm of the mapping T_m.

Similary, if $1 \leq p \leq +\infty$, we say that m is a Schur multiplier on $\mathcal{S}^p$ if for any matrix $[a_{ij}]_{i,j\geq 1}$ in $\mathcal{S}^p$, the matrix $[m_{ij}a_{ij}]_{i,j\geq 1}$ defines an element of $\mathcal{S}^p$.

A simple duality argument shows that if $1 \leq p, p' \leq \infty$ are conjugate numbers, then m is a linear Schur multiplier on $\mathcal{S}^p$ if and only if it is a linear Schur multiplier on $\mathcal{S}^{p'}$.

Moreover the resulting operators T_m have the same norm, that is,

$$\|T_m : \mathcal{S}^p \to \mathcal{S}^p\| = \|T_m : \mathcal{S}^{p'} \to \mathcal{S}^{p'}\|.$$

When $p = 2$, any bounded family $m = \{m_{ij}\}_{i,j\geq 1}$ is a linear Schur multiplier on S^2. Moreover

$$\|M : S^2 \to S^2\| = \sup_{i,j\geq 1} |m_{ij}|$$

in this case (see e.g. [Ara82, Proposition 2.1]).

Note that for $1 < p \neq 2 < +\infty$, there is no description of Schur multipliers on $\mathcal{S}^p$.

There is a well-known characterization of bounded Schur multipliers on $\mathcal{B}(\ell_2)$. This result was stated by Pisier in [Pis96, Theorem 5.1] who refers himself to some earlier work of Grothendieck. This theorem can be extended to the case $\mathcal{B}(\ell_p)$ as follows.

Theorem 2.1. *[Pis96, Theorem 5.10] Let $\phi = (c_{ij})_{i,j\in\mathbb{N}} \subset \mathbb{C}$, $C \geq 0$ be a constant and let $1 \leq p < \infty$. The following are equivalent :*

(i) ϕ is a Schur multiplier on $\mathcal{B}(\ell_p, \ell_p)$ with norm $\leq C$.

(ii) There is a measure space (Ω, μ) and elements $(x_j)_{j\in\mathbb{N}}$ in $L^p(\mu)$ and $(y_i)_{i\in\mathbb{N}}$ in $L^{p'}(\mu)$ such that

$$\forall i, j \in \mathbb{N},\ c_{ij} = \langle x_j, y_i\rangle \text{ and } \sup_i \|y_i\|_{p'} \sup_j \|x_j\|_p \leq C.$$

(iii) The operator $u_\phi : \ell_1 \to \ell_\infty$ which admits $[c_{ij}]$ as its matrix belongs to $\Gamma_p(\ell_1, \ell_\infty)$ and $\gamma_p(u_\phi) \leq C$ (see Remark 1.9 for the notations).

As a consequence of the results established in Subsection 2.3.2, we will characterize more generally Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$ in the case $q \leq p$ in Corollary 2.9 which includes Theorem 2.1. In [Ben77], Bennett gives a necessary and sufficient condition for a family m to be a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$, for all values of p and q, using the theory of absolutely summing operators. Theorem 2.1 above and Corollary 2.9 provide a different type of characterization, which is more explicit and useful.

2.2 Continuous Schur multipliers on $\mathcal{B}(L^2)$

Let (Ω_1, μ_1) and (Ω_2, μ_2) be two σ-finite measure spaces. By the equality (1.16), we have an isometric identification $L^2(\Omega_1 \times \Omega_2) = \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2))$ given by

$$J \in L^2(\Omega_1 \times \Omega_2) \mapsto X_J \in \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2)).$$

Let $\psi \in L^\infty(\Omega_1 \times \Omega_2)$. Thanks to the above identity, we may associate the operator

$$\begin{array}{rccc} R_\psi : & \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2)) & \longrightarrow & \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2)) \\ & X_J & \longmapsto & X_{\psi J} \end{array}$$

whose norm is equal to $\|\psi\|_\infty$. We say that ψ is a *continuous Schur multiplier* if R_ψ extends to a bounded operator (still denoted by)

$$R_\psi \colon \mathcal{K}(L^2(\Omega_1), L^2(\Omega_2)) \longrightarrow \mathcal{B}(L^2(\Omega_1), L^2(\Omega_2)),$$

where $\mathcal{K}(L^2(\Omega_1), L^2(\Omega_2))$ denotes the space of compact operators from $L^2(\Omega_1)$ into $L^2(\Omega_2)$. The density of Hilbert-Schmidt operators in compact operators ensures that this extension is necessarily unique.

The first part of Theorem 2.2 below is a remarkable characterization of continuous Schur multipliers for which we refer e.g. to Spronk [Spr04, Section 3.2]. Peller's characterization of double operator integral mappings which restrict to a bounded operator $\mathcal{S}^1(\mathcal{H}) \to \mathcal{S}^1(\mathcal{H})$ is closely related to this factorization result (see Chapter 4 for a definition of double operator integrals). Indeed, Theorem 2.2(i) below is implicit in [Pel85]. It is also contained in Theorem 2.7 proved in the next section.

For the second part of the next result, recall that by Remark 1.9 and (1.2),

$$\Gamma_2(L^1(\Omega_1), L^\infty(\Omega_2)) \qquad \text{and} \qquad \mathcal{B}\big(\mathcal{K}(L^2(\Omega_1), L^2(\Omega_2)), \mathcal{B}(L^2(\Omega_1), L^2(\Omega_2))\big)$$

are both dual spaces.

We recall that according to the equality (1.5), any element of $\mathcal{B}(L^1(\Omega_1), L^\infty(\Omega_2))$ is an operator u_ψ for some (unique) $\psi \in L^\infty(\Omega_1 \otimes \Omega_2)$.

Theorem 2.2.

(i) [Pel85; Pis96; Spr04] *A function $\psi \in L^\infty(\Omega_1 \times \Omega_2)$ is a continuous Schur multiplier if and only if the operator u_ψ belongs to $\Gamma_2(L^1(\Omega_1), L^\infty(\Omega_2))$, and we have*

$$\gamma_2(u_\psi) = \|R_\psi\|$$

in this case.

(ii) Moreover the isometric embedding

$$\Gamma_2(L^1(\Omega_1), L^\infty(\Omega_2)) \hookrightarrow \mathcal{B}\big(\mathcal{K}(L^2(\Omega_1), L^2(\Omega_2)), \mathcal{B}(L^2(\Omega_1), L^2(\Omega_2))\big)$$

taking any $u_\psi \in \Gamma_2(L^1(\Omega_1), L^\infty(\Omega_2))$ to R_ψ is w^*-continuous.

Proof. Let us prove (2). Let $\psi \in L^\infty(\Omega_1 \times \Omega_2)$ and let $(\psi_\iota)_\iota$ be a net of $L^\infty(\Omega_1 \times \Omega_2)$ such that u_ψ and the operators u_{ψ_ι} belong to $\Gamma_2(L^1(\Omega_1), L^\infty(\Omega_2))$ for any ι, $(u_{\psi_\iota})_\iota$ is a bounded net in the latter space, and $u_{\psi_\iota} \to u_\psi$ in the w^*-topology of $\Gamma_2(L^1(\Omega_1), L^\infty(\Omega_2))$. This implies that $u_{\psi_\iota} \to u_\psi$ in the w^*-topology of $\mathcal{B}(L^1(\Omega_1), L^\infty(\Omega_2))$. According to (1.5), this means that $\psi_\iota \to \psi$ in the w^*-topology of $L^\infty(\Omega_1 \times \Omega_2)$.

Let $\xi, \xi' \in L^2(\Omega_1)$ and $\eta, \eta' \in L^2(\Omega_2)$. For any ι, $R_{\psi_\iota}(\overline{\xi} \otimes \eta)$ is the Hilbert-Schmidt operator associated to the L^2-function $\psi_\iota(\overline{\xi} \otimes \eta)$, hence

$$\big\langle \big[R_{\psi_\iota}(\overline{\xi} \otimes \eta)\big](\xi'), \eta' \big\rangle = \int_{\Omega_1 \times \Omega_2} \psi_\iota(t_1, t_2)\overline{\xi(t_1)}\xi'(t_1)\eta(t_2)\overline{\eta'(t_2)}\, \mathrm{d}\mu_1(t_1)\mathrm{d}\mu_2(t_2)\,.$$

The right-hand side of this equality is the action of $\psi_\iota \in L^\infty(\Omega_1 \times \Omega_2)$ on the L^1-function

$$(t_1, t_2) \mapsto \overline{\xi(t_1)}\xi'(t_1)\eta(t_2)\overline{\eta'(t_2)}.$$

Since $\psi = w^*\text{-}\lim_\iota \psi_\iota$, this implies that

$$\big\langle \big[R_{\psi_\iota}(\overline{\xi} \otimes \eta)\big](\xi'), \eta' \big\rangle \longrightarrow \big\langle \big[R_{\psi}(\overline{\xi} \otimes \eta)\big](\xi'), \eta' \big\rangle.$$

By linearity, this implies that for any finite rank operator $\sigma\colon L^2(\Omega_1)\to L^2(\Omega_2)$, $R_{\psi_\iota}(\sigma)\to R_\psi(\sigma)$ is the weak operator topology of $\mathcal{B}(L^2(\Omega_1),L^2(\Omega_2))$. Since $(u_{\psi_\iota})_\iota$ is a bounded net, $(R_{\psi_\iota})_\iota$ is bounded as well. By the density of finite rank operators in $\mathcal{K}(L^2(\Omega_1),L^2(\Omega_2))$, we deduce that for any σ in the latter space, $R_{\psi_\iota}(\sigma)\to R_\psi(\sigma)$ is the weak operator topology of $\mathcal{B}(L^2(\Omega_1),L^2(\Omega_2))$. Using again the boundedness of $(R_{\psi_\iota})_\iota$, we deduce that $R_{\psi_\iota}(\sigma)\to R_\psi(\sigma)$ in the w^*-topology of $\mathcal{B}\big(L^2(\Omega_1),L^2(\Omega_2)\big)$ for any $\sigma\in\mathcal{K}(L^2(\Omega_1),L^2(\Omega_2))$ and finally that $R_{\psi_\iota}\to R_\psi$ in the w^*-topology of $\mathcal{B}\big(\mathcal{K}(L^2(\Omega_1),L^2(\Omega_2)),\mathcal{B}(L^2(\Omega_1),L^2(\Omega_2))\big)$. $\square$

By Remark 1.20, the embedding of $\Gamma_2(L^1(\Omega_1),L^\infty(\Omega_2))$ into the space

$$\mathcal{B}\big(\mathcal{K}(L^2(\Omega_1),L^2(\Omega_2)),\mathcal{B}(L^2(\Omega_1),L^2(\Omega_2))\big),$$

provided by Theorem 2.2, we obtain a w^*-continuous isometric inclusion

$$L^\infty_\sigma\big(\Omega;\Gamma_2(L^1(\Omega_1),L^\infty(\Omega_2))\big)\ \subset\ L^\infty_\sigma\big(\Omega;\mathcal{B}(\mathcal{K}(L^2(\Omega_1),L^2(\Omega_2)),\mathcal{B}(L^2(\Omega_1),L^2(\Omega_2)))\big). \quad (2.2)$$

2.3 Schur multipliers on $\mathcal{B}(L^p,L^q)$

2.3.1 Definition and connection with the classical Schur multipliers

Let $(\Omega_1,\mathcal{F}_1,\mu_1)$ and $(\Omega_2,\mathcal{F}_2,\mu_2)$ be two σ-finite measure spaces and let $\phi\in L^\infty(\Omega_1\times\Omega_2)$. Let $1\le p,q\le\infty$ and denote by p' and q' their conjugate exponents.
Let

$$T_\phi : L^{p'}(\Omega_1)\otimes L^q(\Omega_2)\to\mathcal{B}(L^p(\Omega_1),L^q(\Omega_2))$$

be defined for any elementary tensor $f\otimes g\in L^{p'}(\Omega_1)\otimes L^q(\Omega_2)$ by

$$[T_\phi(f\otimes g)](h)=\left(\int_{\Omega_1}\phi(s,\cdot)f(s)h(s)\mathrm{d}\mu_1(s)\right)g(\cdot)\in L^q(\Omega_2),$$

for all $h\in L^p(\Omega_1)$.

We have an inclusion

$$L^{p'}(\Omega_1)\otimes L^q(\Omega_2)\subset L^{p'}(\Omega_1,L^q(\Omega_2))$$

given by $f\otimes g\mapsto[s\in\Omega_1\mapsto f(s)g]$. Under this identification, T_ϕ is the multiplication by ϕ. Note that $L^{p'}(\Omega_1,L^q(\Omega_2))$ is invariant by multiplication by an element of $L^\infty(\Omega_1\times\Omega_2)$ and that we have a contractive inclusion

$$L^{p'}(\Omega_1,L^q(\Omega_2))\subset L^{p'}(\Omega_1)\overset{\vee}{\otimes}L^q(\Omega_2).$$

Therefore, T_ϕ is valued is in $L^{p'}(\Omega_1)\overset{\vee}{\otimes}L^q(\Omega_2)$. Using the identification

$$L^{p'}(\Omega_1)\overset{\vee}{\otimes}L^q(\Omega_2)\subset\mathcal{B}(L^p(\Omega_1),L^q(\Omega_2))$$

given by (1.6), we deduce that the elements of $L^{p'}(\Omega_1)\overset{\vee}{\otimes}L^q(\Omega_2)$ are compact operators as limits of finite rank operators for the operator norm.

Definition 2.3. *We say that ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if there exists a constant $C \geq 0$ such that for all $u \in L^{p'}(\Omega_1) \otimes L^q(\Omega_2)$,*

$$\|T_\phi(u)\|_{\mathcal{B}(L^p(\Omega_1),L^q(\Omega_2))} \leq \|u\|_\vee,$$

that is, if T_ϕ extends to a bounded operator

$$T_\phi : L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) \to L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2).$$

In this case, the norm of ϕ is by definition the norm of T_ϕ.

Remark 2.4. *By $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) we denote the space of simple functions on Ω_1 (resp. Ω_2). By density of $\mathcal{E}_1 \otimes \mathcal{E}_2$ in $L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$, T_ϕ extends to a bounded operator from $L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$ into itself if and only if it is bounded on $\mathcal{E}_1 \otimes \mathcal{E}_2$ equipped with the injective tensor norm.*

Assume that $1 < p, q < +\infty$. By (1.7) we have

$$L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) = \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)),$$

so that ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if and only if T_ϕ extends to a bounded operator

$$T_\phi : \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)) \to \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)).$$

In this case, considering the bi-adjoint of T_ϕ, we obtain by (1.8) a w^*-continuous mapping

$$\tilde{T}_\phi : \mathcal{B}(L_p(\Omega_1), L_q(\Omega_2)) \to \mathcal{B}(L_p(\Omega_1), L_q(\Omega_2))$$

which extends T_ϕ. This explains the terminology 'ϕ is a Schur multiplier on $\mathcal{B}(L_p(\Omega_1), L_q(\Omega_2))$'.

Classical Schur multipliers : Assume that $\Omega_1 = \Omega_2 = \mathbb{N}$ and that μ_1 and μ_2 are the counting measures. An element $\phi \in L^\infty(\mathbb{N}^2)$ is given by a family $c = (c_{ij})_{i,j \in \mathbb{N}}$ of complex numbers, where $c_{ij} = \phi(j, i)$. In this situation, the mapping T_ϕ is nothing but the classical Schur multiplier

$$A = [a_{ij}]_{i,j\geq 1} \in \mathcal{B}(\ell_p, \ell_q) \longmapsto [c_{ij}a_{ij}]_{i,j\geq 1}.$$

When this mapping is bounded from $\mathcal{B}(\ell_p, \ell_q)$ into itself, we will denote it by T_c.

Notations : If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $n \in \mathbb{N}^*$, we denote by $\mathcal{A}_{n,\Omega}$ the collection of $n-$tuples $(A_1, \ldots, A_n)$ of pairwise disjoint elements of $\mathcal{F}$ such that

$$\text{for all } 1 \leq i \leq n, 0 < \mu(A_i) < +\infty.$$

If $A = (A_1, \ldots, A_n) \in \mathcal{A}_{n,\Omega}$ and $1 \leq p \leq +\infty$, denote by $S_{A,p}$ the subspace of $L^p(\Omega)$ generated by $\chi_{A_1}, \ldots, \chi_{A_n}$. Then $S_{A,p}$ is $1-$complemented in $L^p(\Omega)$, and a norm one projection from $L^p(\Omega)$ into $S_{A,p}$ is given by the conditional expectation

$$\begin{array}{rrcl} P_{A,p}: & L^p(\Omega) & \longrightarrow & L^p(\Omega). \\ & f & \longmapsto & \displaystyle\sum_{i=1}^{n} \frac{1}{\mu(A_i)} \left(\int_{A_i} f\right) \chi_{A_i} \end{array} \tag{2.3}$$

Note that the mapping

$$\begin{array}{rrcl} \varphi_{A,p}: & S_{A,p} & \longrightarrow & \ell_p^n. \\ & f = \sum_i a_i \chi_{A_i} & \longmapsto & (a_i(\mu_1(A_i))^{1/p})_{i=1}^n \end{array} \tag{2.4}$$

is an isometric isomorphism between $S_{A,p}$ and ℓ_p^n.

Proposition 2.5. *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. The following are equivalent :*

(i) ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$.

(ii) For all $n, m \in \mathbb{N}^*$, for all $A = (A_1, \ldots, A_n) \in \mathcal{A}_{n,\Omega_1}$, $B = (B_1, \ldots, B_m) \in \mathcal{A}_{m,\Omega_2}$, write

$$\phi_{ij} = \frac{1}{\mu_1(A_j)\mu_2(B_i)} \int_{A_j \times B_i} \phi \, d\mu_1 d\mu_2.$$

Then the Schur multipliers on $\mathcal{B}(\ell_p^n, \ell_q^m)$ associated with the families $\phi_{A,B} = (\phi_{ij})$ are uniformly bounded with respect to n, m, A and B.

In this case, $\|T_\phi\| = sup_{n,m,A,B} \|T_{\phi_{A,B}}\| < +\infty$.

Proof. $(i) \Rightarrow (ii)$. Assume first that ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ with $\|T_\phi\| \leq 1$. Let $n, m \in \mathbb{N}^*$, $A = (A_1, \ldots, A_n) \in \mathcal{A}_{n,\Omega_1}$ and $B = (B_1, \ldots, B_m) \in \mathcal{A}_{m,\Omega_2}$. Let $c = \sum_{i,j} c(i,j) e_j \otimes e_i \in \ell_{p'}^n \otimes \ell_q^m \simeq \mathcal{B}(\ell_p^n, \ell_q^m)$.
Let $\varphi_{A,p} : S_{A,p} \to \ell_p^n$ and $\psi_{B,q} : S_{B,q} \to \ell_q^m$ be the isometries defined in (2.4). Then $\tilde{c} := \psi_{B,q}^{-1} \circ c \circ \varphi_{A,p} : S_{A,p} \to S_{B,q}$ satisfies $\|\tilde{c}\| = \|c\|$ and we have

$$\begin{aligned} \tilde{c} &= \sum_{i,j} \frac{c(i,j)}{\mu_1(A_j)^{1/p'} \mu_2(B_i)^{1/q}} \chi_{A_j} \otimes \chi_{B_i} \\ &:= \sum_{i,j} \tilde{c}(i,j) \chi_{A_j} \otimes \chi_{B_i}. \end{aligned}$$

where $\tilde{c}(i,j) = \dfrac{c(i,j)}{\mu_1(A_j)^{1/p'} \mu_2(B_i)^{1/q}}$.
The operator $u := \psi_{B,q} \circ P_{B,q} \circ T_\phi(\tilde{c})_{|S_{A,p}} \circ \varphi_{A,p}^{-1} : \ell_p^n \to \ell_q^m$ satisfies

$$\|u\| \leq \|T_\phi(\tilde{c})\|$$

and by assumption

$$\|T_\phi(\tilde{c})\| \leq \|\tilde{c}\|$$

so that

$$\|u\| \leq \|\tilde{c}\| = \|c\|. \tag{2.5}$$

Let us prove that $u = T_{\phi_{A,B}}(c)$ where $T_{\phi_{A,B}}$ is the Schur multiplier associated with the family (ϕ_{ij}).
Write $u(i,j) := \psi_{B,q} \circ P_{B,q} \circ T_\phi(\chi_{A_j} \otimes \chi_{B_i})_{|S_{A,p}} \circ \varphi_{A,p}^{-1}$. We have

$$u = \sum_{i,j} \tilde{c}(i,j) u(i,j).$$

Let $1 \leq k \leq n$.

$$\begin{aligned}
[u(i,j)](e_k) &= [\psi_{B,q} \circ P_{B,q} \circ T_\phi(\chi_{A_j} \otimes \chi_{B_i})_{|S_{A,p}}] \left(\frac{1}{\mu_1(A_k)^{1/p}} \chi_{A_k} \right) \\
&= \frac{1}{\mu_1(A_k)^{1/p}} [\psi_{B,q} \circ P_{B,q}] \left(\chi_{B_i}(\cdot) \int_{\Omega_1} \phi(s,\cdot) \chi_{A_j}(s) \chi_{A_k}(s) \mathrm{d}\mu_1(s) \right)
\end{aligned}$$

so that $[u(i,j)](e_k) = 0$ if $k \neq j$ and if $k = j$ then

$$\begin{aligned}
[u(i,j)](e_k) &= \frac{1}{\mu_1(A_k)^{1/p}} [\psi_{B,q} \circ P_{B,q}] \left(\chi_{B_i}(\cdot) \int_{A_j} \phi(s,\cdot) \mathrm{d}\mu_1(s) \right) \\
&= \frac{1}{\mu_1(A_k)^{1/p} \mu_2(B_i)} \left(\int_{A_j \times B_i} \phi \right) \psi_q(\chi_{B_i}) \\
&= \frac{1}{\mu_1(A_k)^{1/p} \mu_2(B_i)^{1/q'}} \left(\int_{A_j \times B_i} \phi \right) e_i.
\end{aligned}$$

It follows that

$$\begin{aligned}
u &= \sum_{i,j} \frac{c(i,j)}{\mu_1(A_j)^{1/p'} \mu_2(B_i)^{1/q}} \frac{1}{\mu_1(A_j)^{1/p} \mu_2(B_i)^{1/q'}} \left(\int_{A_j \times B_i} \phi \right) e_j \otimes e_i \\
&= \sum_{i,j} \frac{c(i,j)}{\mu_1(A_j) \mu_2(B_i)} \left(\int_{A_j \times B_i} \phi \right) e_j \otimes e_i \\
&= \sum_{i,j} \phi_{ij} c(i,j) e_j \otimes e_i
\end{aligned}$$

that is, $u = T_{\phi_{A,B}}(c)$. We conclude thanks to the inequality (2.5).

$(ii) \Rightarrow (i)$. Assume now that the assertion (ii) is satisfied and show that ϕ is a Schur multiplier. By Remark 2.4, we just need to show that T_ϕ is bounded on $\mathcal{E}_1 \otimes \mathcal{E}_2$. Let $v \in \mathcal{E}_1 \otimes \mathcal{E}_2$ and write $\alpha = sup_{n,m,A,B} \|T_c\|$. We will show that $\|T_\phi(v)\| \leq \alpha \|v\|$. By density, it is enough to prove that for any $h_1 \in \mathcal{E}_1, h_2 \in \mathcal{E}_2$,

$$| \langle [T_\phi(v)](h_1), h_2 \rangle_{L^q, L^{q'}} | \leq \alpha \|v\|_{\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))} \|h_1\|_{L^p(\Omega_1)} \|h_2\|_{L^{q'}(\Omega_2)}. \tag{2.6}$$

By assumption, there exist $n, m \in \mathbb{N}^*, A = (A_1, \ldots, A_n) \in \mathcal{A}_{n,\Omega_1}, B = (B_1, \ldots, B_m) \in \mathcal{A}_{m,\Omega_2}$ and complex numbers $v(i,j), a_i, b_j$ such that

$$v = \sum_{i,j} v(i,j)\chi_{A_j} \otimes \chi_{B_i}, h_1 = \sum_j a_j \chi_{A_j} \text{ and } h_2 = \sum_i b_i \chi_{B_i}.$$

Equation (2.6) can be rewritten as

$$\left| \sum_{i,j} v(i,j) a_j b_i \left(\int_{A_j \times B_i} \phi \right) \right| \leq \alpha \|v\| \|h_1\|_{L^p(\Omega_1)} \|h_2\|_{L^{q'}(\Omega_2)}. \tag{2.7}$$

Consider $\tilde{v} := \psi_{B,q} \circ v \circ \varphi_{A,p}^{-1} : \ell_p^n \to \ell_q^m$ and $z := \psi_{B,q} \circ P_{B,q} \circ T_\phi(v)_{|S_{A,p}} \circ \phi_{A,p}^{-1} : \ell_p^n \to \ell_q^m$. The computations made in the first part of the proof show that $z = T_m(\tilde{v})$ where m is the family (ϕ_{ij}).
Now, let $x := \varphi_{A,p}(h_1)$ and $y := \psi_{B,q'}(h_2)$. Since T_m is bounded with norm smaller than α we have

$$| \langle [T_m(\tilde{c})](x), y \rangle_{\ell_q^m, \ell_{q'}^m} | \leq \alpha \|\tilde{c}\|_{\mathcal{B}(\ell_p^n, \ell_q^m)} \|x\|_{\ell_p^n} \|y\|_{\ell_{q'}^m}. \tag{2.8}$$

An easy computation shows that the left-hand side on this equality is nothing but the left-hand side of the inequality (2.7). Finally, the right-hand side of the inequalities (2.7) and (2.8) are equal, which concludes the proof.

□

2.3.2 Schur multipliers and factorization

Let p, q be two positive numbers such that $1 \leq q \leq p \leq \infty$. This condition is equivalent to $p, q \in [1, \infty]$ with $\dfrac{1}{q} + \dfrac{1}{p'} \geq 1$, so that we can consider the space $\mathcal{L}_{q,p'}$.

The following results will allow us to give a description of the functions ϕ which are Schur multipliers.

Lemma 2.6. *Let X, Y be Banach spaces and let $E \subset X, F \subset Y$ be $1-$complemented subspaces of X and Y. For any $v \in E \otimes F$, denote by $\tilde{\alpha}'_{q,p'}(v)$ the $\alpha'_{q,p'}$-norm of v as an element of $E \otimes F$ and by $\alpha'_{q,p'}(v)$ the $\alpha'_{q,p'}$-norm of v as an element of $X \otimes Y$. Then*

$$\tilde{\alpha}'_{q,p'}(v) = \alpha'_{q,p'}(v).$$

Proof. The inequality $\tilde{\alpha}'_{q,p'}(v) \geq \alpha'_{q,p'}(v)$ is easy to prove. For the converse inequality, take $v = \sum_k e_k \otimes f_k \in E \otimes F$ such that $\alpha'_{q,p'}(v) < 1$ and show that $\tilde{\alpha}'_{q,p'}(v) < 1$. By assumption, there exists $M \subset X$ and $N \subset Y$ finite dimensional subspaces such that $v \in M \otimes N$ and

$$\alpha'(v, M, N) < 1.$$

By assumption, there exist two norm one projections P and Q respectively from X onto E and from Y onto F. Set $M_1 = P(M) \subset E$ and $N_1 = Q(N) \subset F$. M_1 and N_1 are finite dimensional. Moreover, since $v \in E \otimes F$, it is easy to check that $(P \otimes Q)(v) = v$, where, for all $c = \sum_l a_l \otimes b_l \in X \otimes Y$,

$$(P \otimes Q)(c) = \sum_l P(a_l) \otimes Q(b_l).$$

Thus, $v \in M_1 \otimes N_1$. We will show that $\alpha'_{q,p'}(v, M_1, N_1) < 1$.
Let $z = \sum_{j=1}^m x_j^* \otimes y_j^* \in M_1^* \otimes N_1^*$ be such that $\alpha_{q,p'}(z) < 1$ and show that $|\langle v, z\rangle| \leq \alpha'_{q,p'}(v)$, so that $\alpha'_{q,p'}(v, M_1, N_1) \leq 1$.
Let $1 \leq r \leq \infty$ such that

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{p'} - 1.$$

The condition $\alpha_{q,p'}(z) < 1$ in $M_1^* \otimes N_1^*$ implies that z admits a representation $z = \sum_{j=1}^m \lambda_j m_j^* \otimes n_j^*$ where $m_j^* \in M_1^*, n_j^* \in N_1^*$ and

$$\|(\lambda_j)_j\|_{\ell_r} w_p(m_j^*, M_1^*) w_{q'}(n_j^*, N_1^*) < 1.$$

Set $\tilde{z} := \sum_{j=1}^m \lambda_j P^*(m_j^*) \otimes Q^*(n_j^*)$ in $M^* \otimes N^*$. It is easy to check that

$$w_p(P^*(m_j^*), M^*) \leq w_p(m_j^*, M_1^*) \quad \text{and} \quad w_{q'}(Q^*(n_j^*), N^*) \leq w_{q'}(n_j^*, N_1^*).$$

Therefore, $\alpha_{q,p'}(\tilde{z}, M^*, N^*) < 1$. Then, the condition $\alpha'_{q,p'}(v, M, N) < 1$ implies that

$$|\langle v, \tilde{z}\rangle| \leq \alpha'_{q,p'}(v).$$

Finally, we have

$$\begin{aligned}
\langle v, \tilde{z}\rangle &= \sum_{j,k} \lambda_j \left\langle P^*(m_j^*), e_k\right\rangle \left\langle Q^*(n_j^*), f_k\right\rangle \\
&= \sum_{j,k} \lambda_j \left\langle m_j^*, P(e_k)\right\rangle \left\langle n_j^*, Q(f_k)\right\rangle \\
&= \sum_{j,k} \lambda_j \left\langle m_j^*, e_k\right\rangle \left\langle n_j^*, f_k\right\rangle = \langle v, z\rangle,
\end{aligned}$$

and therefore

$$|\langle v, z\rangle| \leq \alpha'_{q,p'}(v).$$

This proves that $\tilde{\alpha}'_{q,p'}(v) < 1$. □

We recall that if $\phi \in L^\infty(\Omega_1 \times \Omega_2)$, we denote by u_ϕ the mapping

$$\begin{array}{rccl} u_\phi: & L^1(\Omega_1) & \longrightarrow & L^\infty(\Omega_2). \\ & f & \longmapsto & \displaystyle\int_{\Omega_1} \phi(s, \cdot) f(s)\, \mathrm{d}\mu_1(s) \end{array}$$

Theorem 2.7. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be two σ-finite measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. Let $1 \leq q \leq p \leq \infty$. Then ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if and only if the operator u_ϕ belongs to $\mathcal{L}_{q,p'}(L^1(\Omega_1), L^\infty(\Omega_2))$. Moreover,*

$$\|T_\phi\| = L_{q,p'}(u_\phi).$$

Proof. Assume first that T_ϕ extends to a bounded operator

$$T_\phi : L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) \to L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$$

with norm ≤ 1. To prove that $u_\phi \in \mathcal{L}_{q,p'}(L^1(\Omega_1), L^\infty(\Omega_2))$ with $L_{q,p'}(u_\phi) \leq 1$, we have to show that for any $v = \sum_k f_k \otimes g_k \in L^1(\Omega_1) \otimes L^1(\Omega_2)$ with $\alpha'_{q,p'}(v) < 1$ we have

$$|u_\phi(v)| = |\sum_k \langle u_\phi(f_k), g_k \rangle| \leq 1.$$

By density, we can assume that f_k, g_k are simple functions. Hence, with the notations introduced in Section 2.3.1 there exist $n, m \in \mathbb{N}^*, A = (A_1, \ldots, A_n) \in \mathcal{A}_{n,\Omega_1}$ and $B = (B_1, \ldots, B_m) \in \mathcal{A}_{m,\Omega_2}$ such that, for all k, $f_k \in S_{A,1}$ and $g_k \in S_{B,1}$.
By Lemma 2.6, the $\alpha'_{q,p'}$-norm of v as an element of $S_{A,1} \otimes S_{B,1}$ is less than 1.

Let $\varphi_{A,1} : S_{A,1} \to \ell_1^n$ and $\psi_{B,1} : S_{B,1} \to \ell_1^m$ the isomorphisms defined in (2.4). Set $v' = \sum_k \varphi_{A,1}(f_k) \otimes \psi_{B,1}(g_k) \in \ell_1^n \otimes \ell_1^m$. Since $\varphi_{A,1}$ and $\psi_{B,1}$ are isometries, we have $\alpha'_{q,p'}(v') < 1$. Using the identification (1.6), we obtain by (1.11) that v' admits a factorization

$$\begin{array}{ccc} \ell_\infty^n & \xrightarrow{v'} & \ell_1^m \\ {\scriptstyle d_\delta}\downarrow & & \uparrow{\scriptstyle d_\gamma} \\ \ell_p^n & \xrightarrow[c]{} & \ell_q^m \end{array}$$

where $\delta = (\delta_1, \ldots, \delta_n)$, $\gamma = (\gamma_1, \ldots, \gamma_m)$, d_δ and d_γ are the operators of multiplication and

$$\|d_\delta\| = \|\delta\|_{\ell_p} = 1, \|d_\gamma\| = \|\gamma\|_{\ell_{q'}} = 1 \text{ and} \|c\| < 1.$$

This factorization means that

$$v' = \sum_{i=1}^m \sum_{j=1}^n \gamma_i c(i,j) \delta_j e_j \otimes e_i.$$

Therefore, we have

$$\begin{aligned} v &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i c(i,j) \delta_j \, \varphi_{A,1}^{-1}(e_j) \otimes \psi_{B,1}^{-1}(e_i) \\ &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i \frac{c(i,j)}{\mu_1(A_j)\mu_2(B_i)} \delta_j \, \chi_{A_j} \otimes \chi_{B_i}. \end{aligned}$$

We compute

$$\begin{aligned} u_\phi(v) &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i \frac{c(i,j)}{\mu_1(A_j)\mu_2(B_i)} \delta_j \left\langle u_\phi(\chi_{A_j}), \chi_{B_i} \right\rangle \\ &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i \frac{c(i,j)}{\mu_1(A_j)\mu_2(B_i)} \delta_j \left\langle T_\phi(\chi_{A_j} \otimes \chi_{B_i})(\chi_{A_j}), \chi_{B_i} \right\rangle \end{aligned}$$

Define

$$\tilde{c} = \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}(i,j)\chi_{A_j} \otimes \chi_{B_i} \in L^{p'}(\Omega_1) \otimes L^q(\Omega_2),$$

where $\tilde{c}(i,j) = c_{i,j}\mu_1(A_j)^{-1/p'}\mu_2(B_i)^{-1/q}$.
Using the identification (1.6), it is easy to check that we have

$$\tilde{c} = \psi_{B,q}^{-1} \circ c \circ \varphi_{A,p} : S_{A,p} \mapsto L^q(\Omega_2).$$

Therefore,

$$\|\tilde{c}\|_{\vee} = \|c\|.$$

We have

$$\begin{aligned}
u_\phi(v) &= \sum_{i=1}^{m}\sum_{j=1}^{n} \gamma_i \frac{\tilde{c}(i,j)\mu_1(A_j)^{1/p'}\mu_2(B_i)^{1/q}}{\mu_1(A_j)\mu_2(B_i)} \delta_j \left\langle T_\phi(\chi_{A_j} \otimes \chi_{B_i})(\chi_{A_j}), \chi_{B_i} \right\rangle \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \gamma_i \tilde{c}(i,j)\mu_1(A_i)^{-j1/p}\mu_2(B_i)^{-1/q'} \delta_j \left\langle T_\phi(\chi_{A_j} \otimes \chi_{B_i})(\chi_{A_j}), \chi_{B_i} \right\rangle \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \left\langle T_\phi(\tilde{c}(i,j)\chi_{A_j} \otimes \chi_{B_i}) \left(\frac{\delta_j}{\mu_1(A_j)^{1/p}}\chi_{A_j} \right), \frac{\gamma_i}{\mu_2(B_i)^{1/q'}}\chi_{B_i} \right\rangle \\
&= \langle T_\phi(\tilde{c})(f), g \rangle_{L^q(\Omega_2), L^{q'}(\Omega_2)},
\end{aligned}$$

where

$$f = \sum_j \frac{\delta_j}{\mu_1(A_j)^{1/p}}\chi_{A_j} \quad \text{and} \quad g = \sum_i \frac{\gamma_i}{\mu_2(B_i)^{1/q'}}\chi_{B_i}.$$

Since $\|T_\phi\| \leq 1$, we deduce that

$$|u_\phi(v)| \leq \|T_\phi(\tilde{c})\|\|f\|_p\|g\|_{q'} \leq \|\tilde{c}\|\|\delta\|_{\ell_p}\|\gamma\|_{\ell_{q'}} = \|c\| \leq 1.$$

Conversely, assume that $u_\phi \in \mathcal{L}_{q,p'}(L^1(\Omega_1), L^\infty(\Omega_2))$ with $L_{q,p'}(u_\phi) \leq 1$. To prove that ϕ is a Schur multiplier, we will use Proposition 2.5. Let $n, m \in \mathbb{N}^*$, $A = (A_1, \ldots, A_n) \in \mathcal{A}_{n,\Omega_1}$ and $B = (B_1, \ldots, B_m) \in \mathcal{A}_{m,\Omega_2}$. Set

$$\phi_{ij} = \frac{1}{\mu_1(A_j)\mu_2(B_i)} \int_{A_j \times B_i} \phi \, \mathrm{d}\mu_1 \mathrm{d}\mu_2.$$

We want to show that the Schur multiplier on $\mathcal{B}(\ell_p^n, \ell_q^m)$ associated to the family $m = (\phi_{ij})_{i,j}$ has a norm less than 1. To prove that, let $c = \sum_{i,j} c(i,j)e_j \otimes e_i \in \mathcal{B}(\ell_p^n, \ell_q^m)$, $x = (x_j)_{j=1}^n$, $y = (y_i)_{i=1}^m$ in $\mathbb{C}$ be such that $\|c\| \leq 1$, $\|x\|_{\ell_p^n} = 1$, $\|y\|_{\ell_{q'}} = 1$. We have to show that

$$|\langle [T_m(c)](x), y \rangle_{\ell_q^m, \ell_{q'}^m}| \leq 1.$$

This inequality can be rewritten as

$$\left| \sum_{i,j} c(i,j) \frac{x_j y_i}{\mu_1(A_j)\mu_2(B_i)} \left(\int_{A_j \times B_i} \phi \right) \right| \leq 1. \tag{2.9}$$

Let $v = \sum_{i,j} x_j c(i,j) y_i e_j \otimes e_i$. According to (1.11), $\alpha'_{q,p'}(v) \leq 1$. Now, let $\tilde{v} = \sum_{i,j} x_j c(i,j) y_i \varphi_{A,1}^{-1}(e_j) \otimes \psi_{B,1}^{-1}(e_i)$. We have

$$\alpha'_{q,p'}(\tilde{v}) = \alpha'_{q,p'}(v) \leq 1$$

and

$$\tilde{v} = \sum_{i,j} \frac{x_j c(i,j) y_i}{\mu_1(A_j)\mu_2(B_i)} \chi_{A_j} \otimes \chi_{B_i}.$$

By assumption, $L_{q,p'}(u_\phi) \leq 1$, which implies that

$$\begin{aligned} |\langle u_\phi, \tilde{v} \rangle| &= \left| \sum_{i,j} c(i,j) \frac{x_j y_i}{\mu_1(A_j)\mu_2(B_i)} \left(\int_{A_j \times B_i} \phi \right) \right| \\ &\leq \alpha'_{q,p'}(\tilde{v}) \\ &\leq 1, \end{aligned}$$

and this is precisely the inequality (2.9). □

Theorem 1.8 and Remark 2.8 allow us to reformulate the previous theorem. The following two corollaries are generalizations of Theorem 2.1. For the first one, we first need the following remark.

Remark 2.8. *Let $X = L^1(\lambda)$ and $Y = L^1(\nu)$ for some σ-finite measure spaces (Ω_1, λ) and (Ω_2, ν). Consider $T \in \mathcal{B}(L^1(\lambda), L^\infty(\nu))$. By (1.5), there exists $\psi \in L^\infty(\lambda \times \nu)$ such that*

$$T = u_\psi.$$

(See (1.5) for the notation.)

(i) If $1 < q < +\infty$, $L^{q'}(\mu)$ has RNP so by (1.31),

$$\mathcal{B}(L^1(\lambda), L^{q'}(\mu)) = L^\infty(\lambda, L^{q'}(\mu)).$$

It means that if $R \in \mathcal{B}(X, L^{q'}(\mu))$, there exists $a \in L^\infty(\lambda, L^{q'}(\mu))$ such that

$$\forall f \in L^1(\lambda), R(f) = \int_{\Omega_1} f(s) a(s) d\lambda(s).$$

(ii) If $1 < p < +\infty$, then using (1.2), (1.3) and (1.4) we obtain

$$B(L^p(\mu), L^\infty(\nu)) = (L^p(\mu) \overset{\wedge}{\otimes} L^1(\nu))^* = L^\infty(\nu, L^{p'}(\mu)).$$

Thus, if $S \in \mathcal{B}(L^p(\mu), L^\infty(\nu))$, there exists $b \in L^\infty(\nu, L^{p'}(\mu))$ such that

$$\forall g \in L^p(\lambda), S(g)(\cdot) = \langle g, b(\cdot) \rangle.$$

We deduce that if $1 < p, q < +\infty$, there exist $a \in L^\infty(\lambda, L^{q'}(\mu))$ and $b \in L^\infty(\nu, L^{p'}(\mu))$ such that for almost every $(s,t) \in \Omega_1 \times \Omega_2$,

$$\psi(s,t) = \langle a(s), b(t) \rangle.$$

If T satisfies Theorem 1.8, the latter implies that for all $f \in L^1(\lambda)$,

$$T(f) = \int_{\Omega_1} \langle a(s), b(\cdot)\rangle \, f(s) \, ds.$$

Using the same identifications we have for the following cases :

1. *If $q = 1$ and $1 < p < +\infty$, then there exist $a \in L^\infty(\lambda \times \mu)$ and $b \in L^\infty(\nu, L^{p'}(\mu))$ such that for almost every $(s,t) \in \Omega_1 \times \Omega_2$,*

$$\psi(s,t) = \langle a(s,\cdot), b(t)\rangle \, .$$

2. *If $1 < q < +\infty$ and $p = +\infty$, then there exist $a \in L^\infty(\lambda, L^{q'}(\mu))$ and $b \in L^\infty(\nu \times \mu)$ such that for almost every $(s,t) \in \Omega_1 \times \Omega_2$,*

$$\psi(s,t) = \langle a(s), b(t,\cdot)\rangle \, .$$

3. *If $q = 1$ and $p = +\infty$, then there exist $a \in L^\infty(\lambda \times \mu)$ and $b \in L^\infty(\nu \times \mu)$ such that for almost every $(s,t) \in \Omega_1 \times \Omega_2$,*

$$\psi(s,t) = \langle a(s,\cdot), b(t,\cdot)\rangle \, .$$

Corollary 2.9. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be two σ-finite measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. Let $1 \leq q \leq p \leq \infty$. The following statements are equivalent :*

(i) ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$.

(ii) There are a measure space (a probability space when $p \neq q$) (Ω, μ), operators $R \in \mathcal{B}(L^1(\Omega_1), L^p(\mu))$ and $S \in \mathcal{B}(L^q(\mu), L^\infty(\Omega_2)))$ such that $u_\phi = S \circ I \circ R$

$$\begin{array}{ccc} L^1(\Omega_1) & \xrightarrow{u_\phi} & L^\infty(\Omega_2) \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle S} \\ L^p(\mu) & \xhookrightarrow[I]{} & L^q(\mu) \end{array}$$

where I is the inclusion mapping.

In the following cases, (i) and (ii) are equivalent to :
If $1 < q \leq p < +\infty$:

(iii) There are a measure space (a probability space when $p \neq q$) (Ω, μ), $a \in L^\infty(\mu_1, L^p(\mu))$ and $b \in L^\infty(\mu_2, L^{q'}(\mu))$ such that, for almost every $(s,t) \in \Omega_1 \times \Omega_2$,

$$\phi(s,t) = \langle a(s), b(t)\rangle \, .$$

If $1 = q < p < +\infty$:

(iii) There are a probability space (Ω, μ), $a \in L^\infty(\mu_1 \times \mu)$ and $b \in L^\infty(\mu_2, L^{q'}(\mu))$ such that for almost every $(s,t) \in \Omega_1 \times \Omega_2$,

$$\phi(s,t) = \langle a(s,\cdot), b(t)\rangle \, .$$

If $1 < q < +\infty$ *and* $p = +\infty$ *:*

(iii) There are a probability space (Ω, μ)*,* $a \in L^\infty(\mu_1, L^p(\mu))$ *and* $b \in L^\infty(\mu_2 \times \mu)$ *such that for almost every* $(s,t) \in \Omega_1 \times \Omega_2$*,*

$$\phi(s,t) = \langle a(s), b(t, \cdot) \rangle .$$

If $q = 1$ *and* $p = +\infty$ *:*

(iii) There are a probability space (Ω, μ)*,* $a \in L^\infty(\mu_1 \times \mu)$ *and* $b \in L^\infty(\mu_2 \times \mu)$ *such that for almost every* $(s,t) \in \Omega_1 \times \Omega_2$*,*

$$\phi(s,t) = \langle a(s, \cdot), b(t, \cdot) \rangle .$$

In this case, $\|T_\phi\| = \inf \|R\|\|I\|\|S\| = \inf \|a\|\|b\|$.

Remark 2.10. *In the previous corollary, the condition* (ii) *implies that every* $\phi \in L^\infty(\Omega_1 \times \Omega_2)$ *is a Schur multiplier on* $\mathcal{B}(L^1(\Omega_1), L^1(\Omega_2))$ *and on* $\mathcal{B}(L^\infty(\Omega_1), L^\infty(\Omega_2))$*.*

In the discrete case, the previous corollary can be reformulated as follow.

Corollary 2.11. *Let* $\phi = (c_{ij})_{i,j \in \mathbb{N}} \subset \mathbb{C}$*,* $C \geq 0$ *be a constant and let* $1 \leq q \leq p \leq +\infty$*. The following are equivalent :*

(i) ϕ *is a Schur multiplier on* $\mathcal{B}(\ell_p, \ell_q)$ *with norm* $< C$*.*

(ii) There exist a measure space (a probability space when $p \neq q$*)* (Ω, μ) *and two bounded sequences* $(x_j)_j$ *in* $L^p(\mu)$ *and* $(y_i)_i$ *in* $L^{q'}(\mu)$ *such that*

$$\forall i, j \in \mathbb{N}, c_{ij} = \langle x_j, y_i \rangle \ \textit{and} \ \sup_i \|y_i\|_{q'} \sup_j \|x_j\|_p < C.$$

2.3.3 An application : the main triangle projection

Let $m_{ij} = 1$ if $i \leq j$ and $m_{ij} = 0$ otherwise. Let T_m be the Schur multiplier associated with the family $m = (m_{ij})$. For any infinite matrix $A = [a_{ij}]$, $T_m(A)$ is the matrix $[b_{ij}]$ with $b_{ij} = a_{ij}$ if $i \leq j$ and $b_{ij} = 0$ otherwise. For that reason, T_m is called the main triangle projection. Similary, we define the n-th main triangle projection as the Schur multiplier on $\mathcal{M}_n(\mathbb{C})$ associated with the family $m_n = (m^n_{ij})_{1 \leq i,j \leq n}$ where $m^n_{ij} = 1$ if $i \leq j$ and $m^n_{ij} = 0$ otherwise. In [KP70], Kwapień and Pelczyński proved that if $1 \leq q \leq p \leq +\infty, p \neq 1, q \neq +\infty$, there exists a constant $K > 0$ such that for all n,

$$\|T_{m_n} : \mathcal{B}(\ell^n_p, \ell^n_q) \to \mathcal{B}(\ell^n_p, \ell^n_q)\| \geq K \ln(n),$$

and this order of growth is obtained for the Hilbert matrices. Those estimates imply that T_m is not bounded on $\mathcal{B}(\ell_p, \ell_q)$. Bennett proved in [Ben76] that when $1 < p < q < \infty$, T_m is bounded from $\mathcal{B}(\ell_p, \ell_q)$ into itself.

The results obtained in subsection 4.10 allow us to give a very short proof of the unbounded case.

Proposition 2.12. *Let $1 \leq q \leq p \leq +\infty, p \neq 1, q \neq +\infty$. Then T_m is not bounded on $\mathcal{B}(\ell_p, \ell_q)$.*

Proof. Assume that T_m is bounded on $\mathcal{B}(\ell_p, \ell_q)$. By Corollary 2.9, there exist a measure space (Ω, μ), $(a_n)_n \in L^p(\mu)$ and $(b_n)_n \in L^{q'}(\mu)$ two bounded sequences such that, for all $i, j \in \mathbb{N}$,

$$m_{ij} = \langle a_j, b_i \rangle . \tag{2.10}$$

By boundedness, $(a_n)_n$ and $(b_n)_n$ admit an accumluation point $a \in L^p(\mu)$ and $b \in L^{q'}(\mu)$ respectively for the weak-* topology. Fix $i \in \mathbb{N}$. For all $j \geq i$, we have

$$\langle a_i, b_j \rangle = 1$$

so that we get

$$\langle a_i, b \rangle = 1.$$

This equality holds for any i hence

$$\langle a, b \rangle = 1.$$

Now fix $j \in \mathbb{N}$. For all $i > j$ we have

$$\langle a_i, b_j \rangle = 0.$$

From this, we deduce as above that

$$\langle a, b \rangle = 0.$$

We obtained a contradiction so T_m cannot be bounded. □

As a consequence, we have, by Proposition 2.5 :

Corollary 2.13. *Let $1 \leq q \leq p \leq +\infty, p \neq 1, q \neq +\infty$. Let $\Omega_1 = \Omega_2 = \mathbb{R}$ with the Lebesgue measure. Then $\phi \in L^\infty(\mathbb{R}^2)$ defined by*

$$\phi(s,t) := \begin{cases} 1, & \text{if } s+t \geq 0 \\ 0 & \text{if } s+t < 0 \end{cases}, \quad s,t \in \mathbb{R}$$

is not a Schur multiplier on $\mathcal{B}(L^p(\mathbb{R}), L^q(\mathbb{R}))$.

Remark 2.14. *One could wonder whether the results of subsection 4.10 can be extended to the case $1 \leq p < q \leq +\infty$, that is, if the boundedness of T_ϕ on $\mathcal{B}(L^p, L^q)$ implies that u_ϕ has a certain factorization. The fact that if $p < q$ the main triangle projection is bounded tells us that m is a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$. Nevertheless, the argument used in the previous proof shows that m cannot have a factorization like in (2.10). Therefore, the case $p < q$ is more tricky. For the discrete case, one can find in [Ben77, Theorem 4.3] a necessary and sufficient condition for a family $(m_{i,j}) \subset \mathbb{C}$ to be a Schur multiplier, for all values of p and q, using the theory of $q-$absolutely summing operators.*

2.4 Inclusion theorems

In this section, we denote by $\mathcal{M}(p,q)$ the space of Schur multipliers on $\mathcal{B}(\ell_p,\ell_q)$.

First, we recall the inclusions relationships between the spaces $\mathcal{M}(p,q)$. Then we will establish new results as applications of those obtained in Section 4.10.

Theorem 2.15. *[Ben77, Theorem 6.1] Let $p_1 \geq p_2$ and $q_1 \leq q_2$ be given. Then $\mathcal{M}(p_1,q_1) \subset \mathcal{M}(p_2,q_2)$ with equality in the following cases:*

(i) $p_1 = p_2 = 1$,

(ii) $q_1 = q_2 = \infty$,

(iii) $q_2 \leq 2 \leq p_2$,

(iv) $q_2 < p_1 = p_2 < 2$,

(v) $2 < q_1 = q_2 < p_2$.

Let (Ω_1,μ_1) and (Ω_2,μ_2) be two measure spaces. If $\mathcal{M}(p_1,q_1) \subset \mathcal{M}(p_2,q_2)$, then using Proposition 2.5 we have that any Schur multiplier on $\mathcal{B}(L^{p_1}(\Omega_1), L^{q_1}(\Omega_2))$ is a Schur multiplier on $\mathcal{B}(L^{p_2}(\Omega_1), L^{q_2}(\Omega_2))$. Hence, the results in the previous theorem hold true for all the Schur multipliers on $\mathcal{B}(L^p, L^q)$.

In the sequel, we will need the notion of type for a Banach space X, for which we refer e.g. to [AK06]. Let $(\mathcal{E}_i)_{i\in\mathbb{N}}$ be a sequence of independent Rademacher random variables. We have the following definition.

Definition 2.16. *A Banach space X is said to have Rademacher type p (in short, type p) for some $1 \leq p \leq 2$ if there is a constant C such that for every finite set of vectors $(x_i)_{i=n}^n$ in X,*

$$\left(\mathbb{E}\left\|\sum_{i=1}^n \mathcal{E}_i x_i\right\|^p\right)^{1/p} \leq C\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}. \tag{2.11}$$

The smallest constant C for which (2.11) holds is called the type-p constant of X.

We will use the fact that for $1 \leq p \leq 2$, L^p-spaces have type p and if $2 < p < +\infty$, L^p-spaces have type 2 and that those are the best types for infinite dimensional L^p-spaces (see for instance [AK06, Theorem 6.2.14]). We will also use the fact that the type is stable by passing to quotients. Namely, if X has type p and $E \subset X$ is a closed subspace, then X/E has type p.

Proposition 2.17. (i) *If $1 \leq q < p \leq 2$, then*

$$\mathcal{M}(q,1) \nsubseteq \mathcal{M}(p,p).$$

Consequently, for any $1 \leq r \leq q$,

$$\mathcal{M}(q,r) \nsubseteq \mathcal{M}(p,p).$$

(ii) *If $2 \leq p < q \leq r$, then*

$$\mathcal{M}(r,q) \nsubseteq \mathcal{M}(p,p).$$

(iii) If $1 < q < 2 < p < +\infty$ *or* $1 < p < 2 < q < +\infty$, *then*

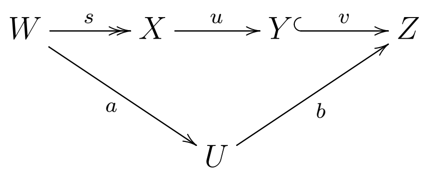
$$\mathcal{M}(q,q) \nsubseteq \mathcal{M}(p,p).$$

To prove this proposition, we will need the following definiton and lemma.

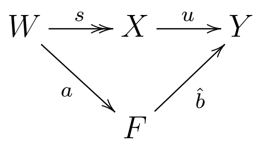
Definition 2.18. *Let* X *and* Y *be Banach spaces,* $u \in \mathcal{B}(X,Y)$ *and* $1 \leq p \leq \infty$. *We say that* $u \in SQ_p(X,Y)$ *if there exists a closed subspace* Z *of a quotient of a* L^p*-space and two operators* $A \in \mathcal{B}(X,Z)$ *and* $B \in \mathcal{B}(Z,Y)$ *such that* $u = BA$.
Then $\|u\|_{SQ_p} = \inf \|A\|\|B\|$ *defines a norm on* $SQ_p(X,Y)$ *and* $(SQ_p(X,Y), \|.\|_{SQ_p})$ *is a Banach space.*

Lemma 2.19. *Let* W, X, Y, Z *be Banach spaces and let* $u \in \mathcal{B}(X,Y), s \in \mathcal{B}(W,X), v \in \mathcal{B}(Y,Z)$ *such that* s *is a quotient map,* v *is a linear isometry and* $vus \in \Gamma_p(W,Z)$. *Then* $u \in SQ_p(X,Y)$.

Proof. By assumption, there exist a L^p-space U and two operators $a \in \mathcal{B}(W,U)$ and $b \in \mathcal{B}(U,Z)$ such that the following diagram commutes



Since v is an isometry, $V := v(Y) \subset Z$ is isometrically isomorphic to Y. Let $\psi : Y \to V$ be the isometric isomorphism induced by v.
Set $F := \{x \in U \text{ such that } b(x) \in V\}$. Since $vus = ba$, we have, for all $w \in W$, $v(us(w)) = b(a(w))$, so that $a(w) \in F$. This implies that $a(W) \subset F$. We still denote by a the mapping $a : W \to F$ and by b the restriction of b to F. Denote by $\hat{b}$ the mapping $\hat{b} = \psi^{-1} \circ b : F \to Y$. Then we have the following commutative diagram



Now, set $E := \overline{a(\ker(s))}$ and let $Q : F \to F/E$ be the canonical mapping. Clearly, $Q \circ a : W \to F/E$ vanishes on $\ker(s)$, so that we have a mapping

$$\widehat{Q \circ a} : W/\ker(s) \to F/E$$

induced by $Q \circ a$.
Since s is a quotient map, we denote by $\widehat{s}$ the isometric isomorphism

$$\hat{s} : W/\ker(s) \to X.$$

Define

$$A = \widehat{Q \circ a} \circ \hat{s}^{-1} : X \to F/E.$$

$\hat{b}$ vanishes on E so that we have a mapping

$$B : F/E \to Y.$$

Finally, it is easy to check that $u = BA$, that is, we have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & Y \\ & {\scriptstyle A}\searrow \quad \nearrow{\scriptstyle B} & \\ & F/E & \end{array}$$

which concludes the proof. □

Remark 2.20. *To prove Lemma 2.19, one can use a result of Kwapień characterizing elements of SQ_p, as follows : a Banach space X is isomorphic to an SQ_q-space if and only if there exists a constant $K \geq 1$ such that for any $n \geq 1$, for any $n \times n$ matrix $[a_{ij}]$ and for any $x_1, \ldots, x_n$ in X,*

$$\left(\sum_i \left\| \sum_j a_{ij} x_j \right\|^q \right)^{1/q} \leq K \| [a_{ij}] : \ell_q^n \to \ell_q^n \| \left(\sum_j \|x_j\|^q \right)^{1/q}.$$

However, the proof presented in here also works if we replace in the statement of the lemma Γ_p (respectively SQ_p) by the space of operators that can be factorized by some Banach space L (respectively by a subspace of a quotient of L).

Proof of Proposition 2.17. (i). Let $\Omega := [0,1]$ and λ be the Lebesgue measure on Ω. Let $I_q : L^q(\lambda) \to L^1(\lambda)$ be the inclusion mapping. By the classical Banach space theory (see [AK06, Theorem 2.3.1] and [AK06, Theorem 2.5.7]) there exist a quotient map $\sigma : \ell_1 \twoheadrightarrow L^q(\lambda)$ and an isometry $J : L^1(\lambda) \hookrightarrow \ell_\infty$. Let $\phi \in \ell_\infty(\mathbb{N}^2)$ be such that

$$u_\phi = J I_q \sigma$$

(by (1.5) any continuous linear map $\ell_1 \to \ell_\infty$ is a certain u_ϕ for $\phi \in L^\infty(\mathbb{N} \times \mathbb{N})$). We have the following factorization

$$\begin{array}{ccc} \ell_1 & \xrightarrow{\quad u_\phi \quad} & \ell_\infty \\ {\scriptstyle \sigma}\downarrow & & \uparrow{\scriptstyle J} \\ L^q(\lambda) & \xhookrightarrow[I_q]{} & L^1(\lambda) \end{array}$$

According to Theorem 2.9, $\phi \in \mathcal{M}(q,1)$.

Assume that $\phi \in \mathcal{M}(p,p)$. Then, again by Theorem 2.9, we have $u_\phi \in \Gamma_p(\ell_1, \ell_\infty)$ and therefore, by Lemma 2.19, there exist an SQ_p-space X and two operators $\alpha \in \mathcal{B}(L^q(\lambda), X)$ and $\beta \in \mathcal{B}(X, L^1(\lambda))$ such that $I_q = \beta\alpha$.
Let $(\mathcal{E}_i)_{i \in \mathbb{N}}$ be a sequence of independant Rademacher random variables. Let $n \in \mathbb{N}^*$ and $f_1, \ldots, f_n \in L^q(\lambda)$.

$$\mathbb{E} \left\| \sum_{j=1}^n \mathcal{E}_j f_j \right\|_{L^1(\lambda)} = \mathbb{E} \left\| \sum_{j=1}^n \mathcal{E}_j \beta\alpha(f_j) \right\|_{L^1(\lambda)} \leq \|\beta\| \mathbb{E} \left\| \sum_{j=1}^n \mathcal{E}_j \alpha(f_j) \right\|_X.$$

But X has type p so there exists a constant $C_1 > 0$ such that

$$\mathbb{E}\left\|\sum_{j=1}^n \mathcal{E}_j f_j\right\|_{L^1(\lambda)} \leq C_1\|\beta\|\left(\sum_{j=1}^n \|\alpha(f_j)\|_X^p\right)^{1/p} \leq C_1\|\beta\|\|\alpha\|\left(\sum_{j=1}^n \|f_j\|_{L^q(\lambda)}^p\right)^{1/p}.$$

By Khintchine inequality, there exists $C_2 > 0$ such that

$$\left\|\left(\sum_{j=1} |f_j|^2\right)^{1/2}\right\|_{L^1(\lambda)} \leq C_2\mathbb{E}\left\|\sum_{j=1}^n \mathcal{E}_j f_j\right\|_{L^1(\lambda)}.$$

Thus, setting $K := C_1C_2\|\alpha\|\|\beta\|$, we obtained the inequality

$$\left\|\left(\sum_{j=1} |f_j|^2\right)^{1/2}\right\|_{L^1(\lambda)} \leq K\left(\sum_{j=1}^n \|f_j\|_{L^q(\lambda)}^p\right)^{1/p}.$$

Let $E_1, \dots, E_n$ be disjoint measurable subsets of $[0,1]$ such that for all $1 \leq j \leq n$, $\lambda(E_j) = \frac{1}{n}$. Set $f_j := \chi_{E_j}$. Then

$$\sum_j |f_j|^2 = 1 \quad \text{and} \quad \|f\|_{L^q(\lambda)} = n^{-1/q}.$$

Hence, applying the previous inequality to the f_j's, we obtain

$$1 \leq Kn^{1/p-1/q}.$$

Since $q < p$, this inequality can't hold for all n, so we obtained a contradiction.

Finally, notice that if $1 \leq r \leq q$, then by Theorem 2.15, $\mathcal{M}(q,1) \subset \mathcal{M}(q,r)$. Thus, $\mathcal{M}(q,r) \not\subseteq \mathcal{M}(p,p)$.

(ii). By Proposition 2.5 and using duality, it is easy to prove that for all $s,t \in [1,\infty]$, ϕ is a Schur multiplier on $\mathcal{B}(\ell_s,\ell_t)$ if and only if $\tilde{\phi}$ is a Schur multiplier on $\mathcal{B}(\ell_{t'},\ell_{s'})$, where $\tilde{\phi}$ is defined for all $i,j \in \mathbb{N}$ by $\tilde{\phi}(i,j) = \phi(j,i)$.
Let $2 \leq p < q \leq r$. Then $1 \leq r' \leq q' < p' \leq 2$. If we assume that $\mathcal{M}(r,q) \subset \mathcal{M}(p,p)$ then the latter implies $\mathcal{M}(q',r') \subset \mathcal{M}(p',p')$, which is, by (i), a contradiction. This proves (ii).

(iii). By duality, it is enough to consider the case $1 < q < 2 < p < +\infty$. Assume that $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$. Using the notations introduced in the proof of (i), let $\sigma : \ell_1 \to \ell_q$ be a quotient map and $J : \ell_q \to \ell_\infty$ be an isometry. Let $\phi \in L^\infty(\mathbb{N}\times\mathbb{N})$ be such that

$$u_\phi = JI_{\ell_q}\sigma,$$

where $I_{\ell_q} : \ell_q \to \ell_q$ is the identity map. Then $\phi \in \mathcal{M}(q,q)$. By assumption, $\phi \in \mathcal{M}(p,p)$. By Lemma 2.19, this implies that $I_{\ell_q} \in SQ_p(\ell_q,\ell_q)$. Clearly, this implies that

ℓ_q is isomorphic to an SQ_p-space. But ℓ_q does not have type 2 and any SQ_p has type 2. This is a contradiction, so $\mathcal{M}(q,q) \not\subseteq \mathcal{M}(p,p)$. □

Theorem 2.21. *We have $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$ if and only if $1 \leq p \leq q \leq 2$ or $2 \leq q \leq p \leq +\infty$.*

Proof. By Proposition 2.17 and duality, we only have to show that when $1 \leq p \leq q \leq 2$, $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$.
We saw in the proof Proposition of 2.17 (iii) that if $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$ then ℓ_q is isomorphic to an SQ_p-space. The converse holds true. Indeed, assume that ℓ_q is isomorphic to an SQ_p-space. Then by approximation, any L^q-space is isomorphic to an SQ_p-space. Hence any element of $\Gamma_q(\ell_1, \ell_\infty)$ factors through an SQ_p-space. By the lifting property of ℓ_1 and the extension property of ℓ_∞, this implies that any element of $\Gamma_q(\ell_1, \ell_\infty)$ factors through an L^p-space, that is $\Gamma_q(\ell_1, \ell_\infty) \subset \Gamma_p(\ell_1, \ell_\infty)$. By Corollary 2.11, this implies that $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$.

Assume that $1 \leq p \leq q \leq 2$. By [AK06, Theorem 6.4.19], there exists an isometry from ℓ_q into an L^p-space, obtained by using $q-$stable processes. Hence, ℓ_q is an SQ_p-space. This concludes the proof.

□

2.5 Perspectives

In Section 2.1, we saw that any bounded family $(m_{ij})_{i,j\in\mathbb{N}}$ of complex numbers is a Schur multiplier on $\mathcal{S}^2(\ell_2)$. Moreover, Theorem 2.1 together with a dual argument give a characterization of Schur multipliers on $\mathcal{S}^1(\ell_2)$ and $\mathcal{B}(\ell_2)$. However, there is no description of Schur multipliers on $\mathcal{S}^p(\ell_2)$ when $1 < p \neq 2 < \infty$. An interesting and difficult problem would be to find an explicit characterization of such multipliers.

The main result of this chapter is a characterization of Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$ in the case when $q \leq p$. As said in Remark 2.14, such characterization cannot hold when $p < q$, because in this case the main triangular projection is bounded on $\mathcal{B}(\ell_p, \ell_q)$. In [Ben77], a necessary and sufficient condition is given for all values of p and q, but it does not allow us to give a handy condition. It is a challenge to find a characterization in the case $p < q$ which is similar to the one given in the case $q \leq p$, that is, a characterization that would imply that the elements of the family (m_{ij}) have a certain form.

Finally, we proved in Section 2.4 some inclusion relationships between the spaces $\mathcal{M}(p,q)$ of Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$. The previous results of inclusions were obtained by Bennett, where he used, as said above, a characterization of Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$ (see [Ben77, Theorem 6.1]). This characterization uses the theory of absolutely summing operators (see the definition e.g. in [Ben77]). The study of such operators reveals that in some particular cases, the space of absolutely summing operators are nothing but the space of bounded operators (see [Ben77, Proposition 5.1]). This is how Bennett could prove his results concerning the inclusions. However, when the two spaces are different, it becomes more complicated to compare the spaces $\mathcal{M}(p,q)$, even in the case $q \leq p$ with the new characterization given in Subsection 2.3.2. Therefore, an open problem is to finish the classification of such spaces. For example, if

$1 < p \leq 2$, do we have

$$\mathcal{M}(p,1) = \mathcal{M}(p,p)?$$

Chapter 3

Bilinear Schur multipliers

In this chapter, we first define bilinear Schur multipliers as bilinear mappings defined on the product of two copies of $\mathcal{S}^2(\ell_2)$. When such mappings are valued in $\mathcal{S}^r(\ell_2)$, we call them bilinear Schur multipliers into $\mathcal{S}^r$. Like in the linear case, any bounded family $M = \{m_{ikj}\}_{i,k,j\geq 1}$ defines a bilinear Schur multiplier into $\mathcal{S}^2$. Similarly, we define continuous bilinear multipliers. In this case, the operators are defined on a product of $\mathcal{S}^2(L^2(\Omega))$-spaces.

The main question of this chapter is to characterize bilinear Schur multipliers into $\mathcal{S}^1$. Theorem 3.4 gives a formula for the norm of those operators in the finite dimensional case. As a consequence, we obtain a characterization of bilinear Schur mulipliers into $\mathcal{S}^1(\ell_2)$ in terms of uniform boundedness of a family of linear Schur multipliers. Following the same ideas, we obtain the main result of this chapter, Theorem 3.8, which describes continuous bilinear Schur multipliers into $\mathcal{S}^1$. A use of Theorem 1.25 allows us to give an explicit characterization of such operators.

3.1 Definition and notations

In this first section, we define bilinear Schur multipliers in the classical case, that is, as mappings defined on $\mathcal{S}^2(\ell_2) \times \mathcal{S}^2(\ell_2)$. The terminology below is adopted from [ER90], where multilinear Schur products are defined and studied in the context of completely bounded maps. Recall that $(E_{ij})_{i,j\in\mathbb{N}}$ denotes the unit matrices of $\mathcal{B}(\ell_2)$.

Definition 3.1. *Let $1 \leq r \leq \infty$. A three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ with entries in $\mathbb{C}$ is said to be a bilinear Schur multiplier into $\mathcal{S}^r$ if the following action*

$$M(A,B) := \sum_{i,j,k\geq 1} m_{ikj} a_{ik} b_{kj} E_{ij}, \quad A = \{a_{ij}\}_{i,j\geq 1}, B = \{b_{ij}\}_{i,j\geq 1} \in \mathcal{S}^2,$$

defines a bounded bilinear operator from $\mathcal{S}^2 \times \mathcal{S}^2$ into $\mathcal{S}^r$.

Of course we can define as well a notion of bilinear Schur multiplier from $\mathcal{S}^p \times \mathcal{S}^q$ into $\mathcal{S}^r$, whenever $1 \leq p,q,r \leq \infty$. The case when $p = q = r = \infty$ was initiated in [ER90] and we will study this case in Chapter 4 in the case of complete boundedness. Let us mention another (easier) case which will be used in Chapter 5.

Lemma 3.2. *A matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ is a bilinear Schur multiplier into $\mathcal{S}^2$ if and only if $\sup_{i,j,k\geq 1} |m_{ikj}| < \infty$. Moreover,*

$$\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2\| = \sup_{i,j,k\geq 1} |m_{ikj}|.$$

Proof. The inequality $\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2\| \leq \sup_{i,j,k\geq 1} |m_{ikj}|$ is achieved by the following computation. Consider $A = \{a_{ik}\}_{i,k\geq 1}$ and $B = \{b_{kj}\}_{k,j\geq 1}$ in $\mathcal{S}^2$. Then applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|M(A,B)\|_2^2 &= \Big\| \sum_{i,j,k\geq 1} m_{ikj} a_{ik} b_{kj} E_{ij} \Big\|_2^2 = \sum_{i,j\geq 1} \Big| \sum_{k\geq 1} m_{ikj} a_{ik} b_{kj} \Big|^2 \\
&\leq \sup_{i,j,k\geq 1} |m_{ikj}|^2 \sum_{i,j\geq 1} \Big(\sum_{k\geq 1} |a_{ik} b_{kj}| \Big)^2 \\
&\leq \sup_{i,j,k\geq 1} |m_{ikj}|^2 \sum_{i,j\geq 1} \sum_{k\geq 1} |a_{ik}|^2 \sum_{k\geq 1} |b_{kj}|^2 \\
&\leq \sup_{i,j,k\geq 1} |m_{ikj}|^2 \|A\|_2^2 \|B\|_2^2.
\end{aligned}$$

The converse inequality is obtained from

$$\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2\| \geq \|M(E_{ik}, E_{kj})\|_2 = |m_{ikj}|,$$

taking the supremum over all $i, j, k \geq 1$. □

3.2 Bilinear Schur multipliers valued in $\mathcal{S}^1$

The aim of this section is to give a criteria when a matrix M is a bilinear Schur multiplier from $\mathcal{S}^2 \times \mathcal{S}^2$ into $\mathcal{S}^1$. The main result is Theorem 3.4 which gives, for $n \in \mathbb{N}$, a formula for the norm of a bilinear Schur multipliers from $\mathcal{S}_n^2 \times \mathcal{S}_n^2$ into $\mathcal{S}_n^1$ in terms of norms of Schur multipliers from M_n into M_n.

We will work with the subspace of $M_n \otimes_{\min} M_n$ spanned by the $E_{rk} \otimes E_{ks}$, for $1 \leq r, k, s \leq n$. The next lemma provides a description of this subspace. We let $(e_1, \ldots, e_n)$ denote the standard basis of ℓ_n^∞.

Lemma 3.3. *The linear mapping $\theta \colon \ell_n^\infty(M_n) \to M_n \otimes_{\min} M_n$ such that*

$$\theta(e_k \otimes E_{rs}) = E_{rk} \otimes E_{ks}, \qquad 1 \leq k, r, s \leq n,$$

is an isometry.

Proof. Take $y = \sum_{k=1}^n e_k \otimes y_k \in \ell_n^\infty(M_n)$, where $y_k = \sum_{r,s=1}^n y_k(r,s) E_{rs}$. From the definition of θ we have

$$\theta(y) = \sum_{r,s,k=1}^n y_k(r,s) E_{rk} \otimes E_{ks}.$$

Recall the isometric isomorphism J_0 given by (1.21). Then

$$J_0\theta(y) = \sum_{r,s,k=1}^{n} y_k(r,s)E_{(r,k),(k,s)}.$$

Let $a = \{a_{rk}\}_{r,k=1}^n, b = \{b_{ls}\}_{l,s=1}^n \in \ell^2_{n^2}$. Then we have

$$\big\langle J_0\theta(y)b, a\big\rangle = \sum_{r,s,k=1}^{n} y_k(r,s)\big\langle E_{(r,k),(k,s)}(b), a\big\rangle = \sum_{r,s,k=1}^{n} y_k(r,s)a_{rk}b_{ks}.$$

Therefore, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\big|\big\langle J_0\theta(y)b, a\big\rangle\big| &\leq \sum_{k=1}^{n}\Big|\sum_{r,s=1}^{n} y_k(r,s)a_{rk}b_{ks}\Big| \\
&\leq \sum_{k=1}^{n}\|y_k\|\Big(\sum_{r=1}^{n}|a_{rk}|^2\Big)^{\frac{1}{2}}\Big(\sum_{s=1}^{n}|b_{ks}|^2\Big)^{\frac{1}{2}} \\
&\leq \max_{1\leq k\leq n}\|y_k\|\sum_{k=1}^{n}\Big(\sum_{r=1}^{n}|a_{rk}|^2\Big)^{\frac{1}{2}}\Big(\sum_{s=1}^{n}|b_{ks}|^2\Big)^{\frac{1}{2}} \\
&\leq \max_{1\leq k\leq n}\|y_k\|\Big(\sum_{k,r=1}^{n}|a_{rk}|^2\Big)^{\frac{1}{2}}\Big(\sum_{k,s=1}^{n}|b_{ks}|^2\Big)^{\frac{1}{2}} \\
&\leq \max_{1\leq k\leq n}\|y_k\|\|a\|_2\|b\|_2.
\end{aligned}$$

It follows that $\|\theta(y)\| \leq \max_{1\leq k\leq n}\|y_k\|$.

Now fix $1 \leq k_0 \leq n$. Take arbitrary $\alpha = \{\alpha_r\}_{r=1}^n$ and $\beta = \{\beta_s\}_{s=1}^n$ in ℓ^2_n. Then define

$$a_{rk} := \begin{cases} \alpha_r, & \text{if } k = k_0 \\ 0 & \text{otherwise} \end{cases}, \qquad b_{ls} := \begin{cases} \beta_s, & \text{if } l = k_0 \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\big\langle J_0\theta(y)b, a\big\rangle = \langle y_{k_0}(\beta), \alpha\rangle$$

and moreover, $\|a\|_2 = \|\alpha\|_2$, $\|b\|_2 = \|\beta\|_2$. Therefore, we have $\|y_{k_0}\| \leq \|\theta(y)\|$. Hence, $\|\theta(y)\| \geq \max_{1\leq k\leq n}\|y_k\|$. □

The following theorem is the main result of this section.

Theorem 3.4. *Let $n \in \mathbb{N}$. Let $M = \{m_{ikj}\}_{i,k,j=1}^n$ be a three-dimensional matrix. For any $1 \leq k \leq n$, let $M(k)$ be the (classical) matrix given by $M(k) = \{m_{ikj}\}_{i,j=1}^n$. We also denote by $M(k) : M_n \to M_n$ the Schur multiplier associated to the family $M(k)$. Then*

$$\big\|M : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\big\| = \sup_{1\leq k\leq n}\big\|M(k) : M_n \to M_n\big\|.$$

Proof. According to the isometric identity (1.1), the bilinear map $M : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n$ induces a linear map $\widetilde{M} \colon \mathcal{S}^2_n\widehat{\otimes}\mathcal{S}^2_n \to \mathcal{S}^1_n$ with $\|M\| = \|\widetilde{M}\|$. Consider

$$T_M = (\widetilde{M}J^{-1})^* \colon M_n \to M_n \otimes_{\min} M_n,$$

where J is given by Lemma 1.15 and where we apply (1.23). This lemma implies that

$$\|T_M\| = \big\|M\colon \mathcal{S}_n^2 \times S_n^2 \to S_n^1\big\|. \tag{3.1}$$

For any $1 \leq r, s \leq n$, we have

$$\begin{aligned}\big\langle T_M(E_{rs}), E_{ij} \otimes E_{kl}\big\rangle &= \big\langle E_{rs}, \widetilde{M} J^{-1}(E_{ij} \otimes E_{kl})\big\rangle \\ &= \big\langle E_{rs}, \widetilde{M}(E_{ik} \otimes E_{jl})\big\rangle \\ &= \begin{cases} m_{ikl}\langle E_{rs}, E_{il}\rangle, & \text{if } k = j \\ 0 & \text{otherwise}\end{cases} \\ &= \begin{cases} m_{ikl}, & \text{if } k = j,\ r = i,\ s = l \\ 0 & \text{otherwise}\end{cases},\end{aligned}$$

for all $1 \leq i, j, k, l \leq n$. Hence

$$T_M(E_{rs}) = \sum_{k=1}^{n} m_{rks} E_{rk} \otimes E_{ks}.$$

This shows that T_M maps into the range of the operator θ introduced in Lemma 3.3 and that

$$T_M(E_{rs}) = \sum_{k=1}^{n} m_{rks}\, \theta(e_k \otimes E_{rs}).$$

By linearity this implies that for any $C \in M_n$,

$$T_M(C) = \theta\Bigg(\sum_{k=1}^{n} e_k \otimes [M(k)](C)\Bigg).$$

Appyling Lemma 3.3, we deduce that

$$\|T_M(C)\| = \max_k \big\|[M(k)](C)\big\|, \qquad C \in M_n.$$

From this identity we obtain that $\|T_M\| = \max_k \|M(k)\|$. Combining with (3.1) we obtain the desired identity $\|M\| = \max_k \|M(k)\|$. $\square$

For the sake of completeness we give an infinite dimensional version of the previous theorem.

Theorem 3.5. *A three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ is a bilinear Schur multiplier into $\mathcal{S}^1$ if and only if the matrix $M(k) = \{m_{ikj}\}_{i,j\geq 1}$ is a linear Schur multiplier on S^∞ for every $k \geq 1$ and $\sup_{k\geq 1} \|M(k) : S^\infty \to S^\infty\| < \infty$. Moreover,*

$$\big\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\big\| = \sup_{k\geq 1} \big\|M(k) : S^\infty \to \mathcal{S}^\infty\big\|$$

in this case.

Proof. Consider a three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ and set $M(k) = \{m_{ikj}\}_{i,j\geq 1}$. For any $n \geq 1$, let

$$M_{(n)} = \{m_{ikj}\}_{1\leq i,j\leq n} \quad \text{and} \quad M_{(n)}(k) = \{m_{ikj}\}_{1\leq i,k,j\leq n}$$

be the standard truncations of these matrices.

We may identify $\mathcal{S}^2_n$ (respectively $\mathcal{S}^\infty_n$) with the subspace of $\mathcal{S}^2$ (respectively $\mathcal{S}^\infty$) spanned by $\{E_{ij} \,:\, 1 \leq i,j \leq n\}$. Then the union $\cup_{n\geq 1}\mathcal{S}^2_n$ is dense in $\mathcal{S}^2$. Hence by a standard density argument, M is a bilinear Schur multiplier into $\mathcal{S}^1$ if and only if $\sup_{n\geq 1}\|M_{(n)} : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\| < \infty$, and in this case

$$\big\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\big\| = \sup_{n\geq 1}\big\|M_{(n)} : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\big\|.$$

Likewise $\cup_{n\geq 1}\mathcal{S}^\infty_n$ is dense in the space $\mathcal{S}^\infty$ of all compact operators, for any $k \geq 1$ $M(k)$ is a linear Schur multiplier on S^∞ if and only if $\sup_{n\geq 1}\|M_{(n)}(k) : \mathcal{S}^\infty_n \to \mathcal{S}^\infty_n\| < \infty$, and

$$\big\|M(k) : \mathcal{S}^\infty \to \mathcal{S}^\infty\big\| = \sup_{n\geq 1}\big\|M_{(n)}(k) : \mathcal{S}^\infty_n \to \mathcal{S}^\infty_n\big\|.$$

in this case.

Combining the above two approximation results with Theorem 3.4, we obtain the result. □

Theorem 3.5 together with Theorem 2.1 yield the following result.

Corollary 3.6. *A three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ is a bilinear Schur multiplier into $\mathcal{S}^1$ if and only if there exist a Hilbert space E and two bounded families $(\xi_{ik})_{i,k\geq 1}$ and $(\eta_{jk})_{j,k\geq 1}$ in E such that*

$$m_{ikj} = \langle \xi_{ik}, \eta_{jk}\rangle, \qquad i,k,j \geq 1.$$

Moreover

$$\big\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\big\| = \inf\big\{\sup_{i,k}\|\xi_{ik}\| \sup_{j,k}\|\eta_{jk}\|\big\},$$

where the infimum runs over all possible such factorizations.

3.3 Continuous bilinear Schur multipliers

In this section, we first define and give few properties of continuous bilinear Schur multipliers. Those mappings are defined on a product of $\mathcal{S}^2(L^2(\Omega))-$spaces. When $\Omega = \mathbb{N}$ with the counting measure, the definition is nothing but the one given in Section 3.1. The main result is Theorem 3.8 which gives a necessary and sufficient condition for a continuous bilinear Schur multiplier to be valued in $\mathcal{S}^1$. This result is the continuous analogue of Theorem 3.5 and it will play an important role in Chapter 4.

3.3.1 Definition

Let $(\Omega_1,\mu_1), (\Omega_2,\mu_2)$ and (Ω_3,μ_3) be three σ-finite measure spaces, and let $\phi \in L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$. For any $J \in L^2(\Omega_1 \times \Omega_2)$ and $K \in L^2(\Omega_2 \times \Omega_3)$, the function

$$\Lambda(\phi)(J,K) \colon (t_1,t_3) \mapsto \int_{\Omega_2} \phi(t_1,t_2,t_3)J(t_1,t_2)K(t_2,t_3)\,\mathrm{d}\mu_2(t_2)$$

is a well-defined element of $L^2(\Omega_1 \times \Omega_3)$ with L^2-norm less than $\|\phi\|_\infty \|J\|_2 \|K\|_2$. Indeed, by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\int_{\Omega_1\times\Omega_3} \left(\int_{\Omega_2} |\phi(t_1,t_2,t_3)J(t_1,t_2)K(t_2,t_3)|\mathrm{d}\mu_2(t_2)\right)^2 \mathrm{d}\mu_1(t_1)\mathrm{d}\mu_3(t_3)\\
&\quad\leq \|\phi\|_\infty^2 \int_{\Omega_1\times\Omega_3} \left(\int_{\Omega_2} |J(t_1,t_2)K(t_2,t_3)|\mathrm{d}\mu_2(t_2)\right)^2 \mathrm{d}\mu_1(t_1)\mathrm{d}\mu_3(t_3)\\
&\quad\leq \|\phi\|_\infty^2 \int_{\Omega_1\times\Omega_3} \left(\int_{\Omega_2} |J(t_1,t_2)|^2\mathrm{d}\mu_2(t_2)\right)\left(\int_{\Omega_2} |K(t_2,t_3)|^2\mathrm{d}\mu_2(t_2)\right) \mathrm{d}\mu_1(t_1)\mathrm{d}\mu_3(t_3)\\
&\quad\leq \|\phi\|_\infty^2 \left(\int_{\Omega_1\times\Omega_2} |J(t_1,t_2)|^2\mathrm{d}\mu_1(t_1)\mathrm{d}\mu_2(t_2)\right)\left(\int_{\Omega_2\times\Omega_3} |K(t_2,t_3)|^2\mathrm{d}\mu_2(t_2)\mathrm{d}\mu_3(t_3)\right).
\end{aligned}$$

Thus $\Lambda(\phi)$ is a bounded bilinear map from $L^2(\Omega_1 \times \Omega_2) \times L^2(\Omega_2 \times \Omega_3)$ into $L^2(\Omega_1 \times \Omega_3)$. By the isometric identification between $L^2(\Omega_1 \times \Omega_2)$ and $\mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2))$ given by (1.16), and their analogues for (Ω_2,Ω_3) and (Ω_1,Ω_3), we may consider that we actually have a bounded bilinear map

$$\Lambda(\phi) \colon \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2)) \times \mathcal{S}^2(L^2(\Omega_2), L^2(\Omega_3)) \longrightarrow \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_3)).$$

We call $\Lambda(\phi)$ a *continuous bilinear Schur multiplier*.

Let $E(\Omega_1,\Omega_2,\Omega_3) = \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2)) \overset{\wedge}{\otimes} \mathcal{S}^2(L^2(\Omega_2), L^2(\Omega_3)) \overset{\wedge}{\otimes} \mathcal{S}^2(L^2(\Omega_3), L^2(\Omega_1))$. By (1.1), (1.2) and (1.12), we have isometric identifications,

$$E(\Omega_1,\Omega_2,\Omega_3)^* = \mathcal{B}_2(\mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2)) \times \mathcal{S}^2(L^2(\Omega_2), L^2(\Omega_3)), \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_3)))$$

for the duality pairing given by

$$\big\langle T, X \otimes Y \otimes Z \big\rangle = \mathrm{tr}\big(T(X,Y)Z\big)$$

for any bounded bilinear $T\colon \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2)) \times \mathcal{S}^2(L^2(\Omega_2), L^2(\Omega_3)) \to \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_3))$ and for any $X \in \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2))$, $Y \in \mathcal{S}^2(L^2(\Omega_2), L^2(\Omega_3))$ and $Z \in \mathcal{S}^2(L^2(\Omega_3), L^2(\Omega_1))$.

Proposition 3.7. *The mapping*

$$\Lambda \colon L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3) \longrightarrow E(\Omega_1,\Omega_2,\Omega_3)^*$$

defined above is a w^*-continuous isometry.

Proof. Write $E = E(\Omega_1, \Omega_2, \Omega_3)$ for simplicity. Consider three functions $J \in L^2(\Omega_1 \times \Omega_2)$, $K \in L^2(\Omega_2 \times \Omega_3)$ and $L \in L^2(\Omega_3 \times \Omega_1)$. It is easy to check that the function

$$\varphi\colon (t_1, t_2, t_3) \mapsto J(t_1, t_2)K(t_2, t_3)L(t_3, t_1)$$

belongs to $L^1(\Omega_1 \times \Omega_2 \times \Omega_3)$. Further if $X_J \in \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2))$, $Y_K \in \mathcal{S}^2(L^2(\Omega_2), L^2(\Omega_3))$ and $Z_L \in \mathcal{S}^2(L^2(\Omega_3), L^2(\Omega_1))$ denote the Hilbert-Schmidt operators associated with J, K and L, respectively, then it follows from above that

$$\big\langle \Lambda(\phi), X_J \otimes Y_K \otimes Z_L \big\rangle_{E^*,E} \,=\, \int_{\Omega_1 \times \Omega_2 \times \Omega_3} \phi\varphi \,=\, \langle \phi, \varphi \rangle_{L^\infty, L^1}$$

for any $\phi \in L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$. This readily implies that Λ is w^*-continuous.

We already showed that Λ is a contraction, let us now prove that it is an isometry. Let $\phi \in L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$, with $\|\phi\|_\infty > 1$. We aim at showing that $\|\Lambda(\phi)\|_{E^*} > 1$. There exist a function $\varphi \in L^1(\Omega_1 \times \Omega_2 \times \Omega_3)$ such that $\|\varphi\|_1 = 1$ and $\langle \phi, \varphi \rangle_{L^\infty, L^1} > 1$. By the density of simple functions in L^1, we may assume that

$$\varphi = \sum_{i,j,k} m_{ijk}\, \chi_{F_i^1} \otimes \chi_{F_j^2} \otimes \chi_{F_k^3},$$

where $(F_i^1)_i$ (respectively $(F_j^2)_j$ and $(F_k^3)_k$) is a finite family of pairwise disjoint measurable subsets of Ω_1 (respectively of Ω_2 and Ω_3) and $m_{ijk} \in \mathbb{C}$ for any i, j, k. Let $\psi \in E$ be defined by

$$\psi = \sum_{i,j,k} m_{ijk} \big(\chi_{F_i^1} \otimes \chi_{F_j^2}\big) \otimes \big(\chi_{F_j^2} \otimes \chi_{F_k^3}\big) \otimes \big(\chi_{F_k^3} \otimes \chi_{F_i^1}\big).$$

For any i, j, k, we have

$$\begin{aligned} &\big\langle \Lambda(\phi), \big(\chi_{F_i^1} \otimes \chi_{F_j^2}\big) \otimes \big(\chi_{F_j^2} \otimes \chi_{F_k^3}\big) \otimes \big(\chi_{F_k^3} \otimes \chi_{F_i^1}\big) \big\rangle_{E^*,E} \\ &\qquad = \int_{\Omega_1 \times \Omega_2 \times \Omega_3} \phi(t_1, t_2, t_3) \chi_{F_i^1}(t_1) \chi_{F_j^2}(t_2) \chi_{F_k^3}(t_3)\, \mathrm{d}\mu_1(t_1) \mathrm{d}\mu_2(t_2) \mathrm{d}\mu_3(t_3)\,. \end{aligned}$$

This implies that

$$\langle \Lambda(\phi), \psi \rangle_{E^*,E} \,=\, \langle \phi, \varphi \rangle_{L^\infty, L^1},$$

and hence that $\langle \Lambda(\phi), \psi \rangle_{E^*,E} > 1$. Now observe that by the definition of the projective tensor product we have

$$\|\psi\|_E \leq \sum_{i,j,k} |m_{ijk}| \|\chi_{F_i^1} \otimes \chi_{F_j^2}\|_2 \|\chi_{F_j^2} \otimes \chi_{F_k^3}\|_2 \|\chi_{F_k^3} \otimes \chi_{F_j^2}\|_2.$$

Moreover,

$$\|\chi_{F_i^1} \otimes \chi_{F_j^2}\|_2 = \|\chi_{F_i^1}\|_2 \|\chi_{F_j^2}\|_2 = \lambda_1(F_i^1)^{\frac{1}{2}} \lambda_2(F_j^2)^{\frac{1}{2}}.$$

Likewise, $\|\chi_{F_j^2} \otimes \chi_{F_k^3}\|_2 = \lambda_2(F_j^2)^{\frac{1}{2}} \lambda_3(F_k^3)^{\frac{1}{2}}$ and $\|\chi_{F_k^3} \otimes \chi_{F_j^2}\|_2 = \lambda_3(F_k^3)^{\frac{1}{2}} \lambda_1(F_i^1)^{\frac{1}{2}}$. We deduce that

$$\|\psi\|_E \leq \sum_{i,j,k} |m_{ijk}| \lambda_1(F_j^2) \lambda_2(F_j^2) \lambda_3(F_k^3).$$

The right-hand side of this inequality is nothing but the L^1-norm of φ. Thus we have proved that $\|\psi\|_E \leq \|\varphi\|_1 = 1$. This implies that $\|\Lambda(\phi)\|_{E^*} > 1$ as expected. □

3.3.2 $\mathcal{S}^1$-boundedness

In this section, we will determine which functions $\phi \in L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$ are such that $\Lambda(\phi)$ maps $\mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2)) \times \mathcal{S}^2(L^2(\Omega_2), L^2(\Omega_3))$ into $\mathcal{S}^1(L^2(\Omega_1), L^2(\Omega_3))$.

Theorem 3.8. *Let $(\Omega_1, \mu_1), (\Omega_2, \mu_2)$ and (Ω_3, μ_3) be measure spaces and let $H_i = L^2(\Omega_i), i = 1, 2, 3$. Let $\phi \in L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$. The following are equivalent:*

(i) $\Lambda(\phi) \in \mathcal{B}_2(\mathcal{S}^2(H_1, H_2) \times \mathcal{S}^2(H_2, H_3), \mathcal{S}^1(H_1, H_3))$.

(ii) There exist a Hilbert space H and two functions

$$a \in L^\infty(\Omega_1 \times \Omega_2; H) \qquad \text{and} \qquad b \in L^\infty(\Omega_2 \times \Omega_3; H)$$

such that

$$\phi(t_1, t_2, t_3) = \langle a(t_1, t_2), b(t_2, t_3)\rangle$$

for a.-e. $(t_1, t_2, t_3) \in \Omega_1 \times \Omega_2 \times \Omega_3$.

In this case

$$\|\Lambda(\phi) : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\| = \inf \|a\|_\infty \|b\|_\infty.$$

Proof. Proof of (i) $\Rightarrow$ (ii)

Assume that $\Lambda(\phi) \in \mathcal{B}_2(\mathcal{S}^2(H_1, H_2) \times \mathcal{S}^2(H_2, H_3), \mathcal{S}^1(H_1, H_3))$.

By the equalities $\mathcal{S}^2(H_1, H_2) = \mathcal{S}^2(H_2, H_1)$ and $\mathcal{S}^2(H_2, H_3) = \mathcal{S}^2(H_3, H_2)$ which are consequences of (1.16), we may asssume that $\Lambda(\phi)$ is a bounded bilinear mapping from $\mathcal{S}^2(H_2, H_1) \times \mathcal{S}^2(H_3, H_2)$ into $\mathcal{S}^1(H_3, H_1)$.

According to the identification

$$\mathcal{B}_2(\mathcal{S}^2 \times \mathcal{S}^2, \mathcal{S}^2) = \mathcal{B}(\mathcal{S}^2 \overset{\wedge}{\otimes} \mathcal{S}^2, \mathcal{S}^2)$$

provided by (1.1), we may regard $\Lambda(\phi)$ as a bounded linear operator

$$\Lambda(\phi) \colon \mathcal{S}^2(H_2, H_1) \overset{\wedge}{\otimes} \mathcal{S}^2(H_3, H_2) \longrightarrow \mathcal{S}^1(H_3, H_1).$$

Let

$$\Phi \colon \mathcal{S}^2(H_2, H_1) \overset{\wedge}{\otimes} \mathcal{S}^2(H_3, H_2) \longrightarrow \mathcal{S}^1(H_2) \overset{\gamma}{\otimes} \mathcal{S}^1(H_3, H_1)$$

be the isomorphism given by Lemma 1.14 (where we naturally identify H_2 with its conjugate space $\overline{H_2}$). Let $w = \Lambda(\phi) \circ \Phi^{-1}$ be the composition map. By the identifications (1.12) and (1.20) given by trace duality, its adjoint map w^* from $\mathcal{S}^1(H_3, H_1)^*$ into $\left(\mathcal{S}^1(H_2) \overset{\gamma}{\otimes} \mathcal{S}^1(H_3, H_1)\right)^*$ can be regarded as a map

$$v \colon \mathcal{B}(H_1, H_3) \longrightarrow \mathcal{B}(H_2) \overline{\otimes} \mathcal{B}(H_1, H_3).$$

We consider the inclusion

$$L^\infty(\mu_2) \subset \mathcal{B}(H_2)$$

obtained by identifying any element of $L^\infty(\mu_2)$ with its associated multiplication operator $L^2(\mu_2) \to L^2(\mu_2)$. We shall now analyse v to get to property (3.2) below.

Take any $c, \xi \in H_1$, $c', d' \in H_2$ and $d, \eta \in H_3$. Regard $(d' \otimes c') \otimes (d \otimes c)$ as an element of $\mathcal{S}^1(H_2) \otimes \mathcal{S}^1(H_3, H_1)$. Then

$$\Phi^{-1}\big((d' \otimes c') \otimes (d \otimes c)\big) = (c' \otimes c) \otimes (d \otimes d'),$$

regarded as an element of $\mathcal{S}^2(H_2, H_1) \otimes \mathcal{S}^2(H_3, H_2)$. Consider $\xi \otimes \eta$ as an element of $\mathcal{B}(H_1, H_3)$. Then

$$\begin{aligned}
&\big\langle v(\xi \otimes \eta), (d' \otimes c') \otimes (d \otimes c)\big\rangle \\
&\quad = \big\langle \xi \otimes \eta, w\big((d' \otimes c') \otimes (d \otimes c)\big)\big\rangle \\
&\quad = \big\langle \xi \otimes \eta, \Lambda(\phi)\big((c' \otimes c) \otimes (d \otimes d')\big)\big\rangle_{\mathcal{B}(H_1,H_3),\mathcal{S}^1(H_3,H_1)} \\
&\quad = \int_{\Omega_1 \times \Omega_2 \times \Omega_3} \phi(t_1, t_2, t_3)\xi(t_1)\eta(t_3)c'(t_2)d'(t_2)c(t_1)d(t_3)\, \mathrm{d}\mu_1(t_1)\mathrm{d}\mu_2(t_2)\mathrm{d}\mu_3(t_3)\,.
\end{aligned}$$

For ξ, η, c, d as above, consider

$$S = \big\langle v(\xi \otimes \eta), \bullet \otimes (d \otimes c)\big\rangle \in \mathcal{B}(H_2).$$

Then the above calculation shows that $S\colon L^2(\mu_2) \to L^2(\mu_2)$ is the multiplication operator associated to the function

$$t_2 \mapsto \int_{\Omega_1 \times \Omega_3} \phi(t_1, t_2, t_3)\xi(t_1)\eta(t_3)c(t_1)d(t_3)\, \mathrm{d}\mu_1(t_1)\mathrm{d}\mu_3(t_3)\,.$$

Thus S belongs to $L^\infty(\mu_2)$.

This implies that for any $(\xi, \eta) \in H_1 \times H_3$, $v(\xi \otimes \eta)$ belongs to the space $L^\infty(\mu_2)\overline{\otimes}\mathcal{B}(H_1, H_3)$. Since v is w^*-continuous and $H_1 \otimes H_3$ is w^*-dense in $\mathcal{B}(H_1, H_3)$, this implies that

$$v\big(\mathcal{B}(H_1, H_3)\big) \subset L^\infty(\mu_2)\overline{\otimes}\mathcal{B}(H_1, H_3). \tag{3.2}$$

Consider now the restriction $v_0 = v_{|\mathcal{K}(H_1,H_3)}$ of v to the subspace $\mathcal{K}(H_1, H_3)$ of compact operators from H_1 into H_3. By Lemma 1.19 and (3.2), we may write

$$v_0\colon \mathcal{K}(H_1, H_3) \longrightarrow L^\infty_\sigma(\mu_2; \mathcal{B}(H_1, H_3)).$$

Corollary 1.17 provides an identification

$$\mathcal{B}\big(\mathcal{K}(H_1, H_3), L^\infty_\sigma(\mu_2; \mathcal{B}(H_1, H_3))\big) = L^\infty_\sigma\big(\mu_2; B(\mathcal{K}(H_1, H_3), \mathcal{B}(H_1, H_3))\big).$$

Let $\widetilde{\phi} \in L^\infty_\sigma\big(\mu_2; B(\mathcal{K}(H_1, H_3), \mathcal{B}(H_1, H_3))\big)$ be corresponding to v_0 in this identification. Then by the preceding computation we have that for any $c, \xi \in H_1$ and $d, \eta \in H_3$,

$$\big\langle \big[\widetilde{\phi}(t_2)\big](\xi \otimes \eta), d \otimes c\big\rangle = \int_{\Omega_1 \times \Omega_3} \phi(t_1, t_2, t_3)\xi(t_1)\eta(t_3)c(t_1)d(t_3)\, \mathrm{d}\mu_1(t_1)\mathrm{d}\mu_3(t_3)$$

for a.e. t_2 in Ω_2.

Following Subsection 2.2, for any $J \in L^2(\Omega_1 \times \Omega_3)$, we let $X_J \in \mathcal{S}^2(H_1, H_3)$ be the Hilbert-Schmidt operator with kernel J. Then the above formula shows that for $J = \xi \otimes \eta$, we have

$$\big[\widetilde{\phi}(t_2)\big](X_J) = X_{\phi(\cdot,t_2,\cdot)J} \qquad \text{for a.e. } t_2. \tag{3.3}$$

By density of $H_1 \otimes H_3$ in $L^2(\Omega_1 \times \Omega_3)$, we deduce that (3.3) holds true for any $J \in L^2(\Omega_1 \times \Omega_3)$. This means that for a.e. t_2, $\phi(\cdot\,, t_2, \cdot\,)$, regarded as an element of $L^\infty(\Omega_1 \times \Omega_3)$, is a continuous Schur multiplier, whose corresponding operator is

$$\widetilde{\phi}(t_2) = R_{\phi(\cdot,t_2,\cdot)} \colon \mathcal{K}(L^2(\Omega_1), L^2(\Omega_3)) \longrightarrow \mathcal{B}(L^2(\Omega_1), L^2(\Omega_3)).$$

This shows two things. First, $\widetilde{\phi}$ belongs to $L^\infty_\sigma\big(\mu_2; \Gamma_2(L^1(\Omega_1), L^\infty(\Omega_3))\big)$ regarded as a subspace of $L^\infty_\sigma\big(\mu_2; \mathcal{B}(\mathcal{K}(H_1, H_3), \mathcal{B}(H_1, H_3))\big)$ by (2.2). Second, the element of $L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$ corresponding to $\widetilde{\phi}$ through the inclusion (1.46) is the function ϕ itself. Thus we have proved that $\phi \in L^\infty_\sigma\big(\mu_2; \Gamma_2(L^1(\Omega_1), L^\infty(\Omega_3))\big)$. Hence, applying Theorem 1.25, we obtain the factorization given in (ii). Moreover, by the same theorem

$$\|\phi\|_{\infty,\Gamma_2} = \inf \|a\|_\infty \|b\|_\infty.$$

Hence, by the above reasoning, we obtain

$$\inf \|a\|_\infty \|b\|_\infty = \|\phi\|_{\infty,\Gamma_2} \leq \big\|\Lambda(\phi) \colon \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\big\|.$$

Proof of (ii) $\Rightarrow$ (i)

Assume that ϕ has the factorization given in (ii). Let $J \in \mathcal{S}^2(H_1, H_2)$ and $K \in \mathcal{S}^2(H_2, H_3)$ identified with elements of $L^2(\Omega_1 \times \Omega_2)$ and $L^2(\Omega_2 \times \Omega_3)$ (see (1.16)). We have, for almost every $(t_1, t_3) \in \Omega_1 \times \Omega_3$,

$$\begin{aligned}\Lambda(\phi)(J, K)(t_1, t_3) &= \int_{\Omega_2} \langle a(t_1, t_2), b(t_2, t_3)\rangle\, J(t_1, t_2) K(t_2, t_3) \mathrm{d}\mu_2(t_2)\\ &= \int_{\Omega_2} \Big\langle \tilde{J}(t_1, t_2), \tilde{K}(t_2, t_3)\Big\rangle\, \mathrm{d}\mu_2(t_2)\end{aligned}$$

where $\tilde{J}(t_1, t_2) = J(t_1, t_2) a(t_1, t_2)$ and $\tilde{K}(t_2, t_3) = \overline{K}(t_2, t_3) b(t_2, t_3)$. Then $\tilde{J} \in L^2(\Omega_1 \times \Omega_2, H)$ and $\tilde{K} \in L^2(\Omega_2 \times \Omega_3, H)$ and we have the estimates

$$\|\tilde{J}\|_2 \leq \|a\|_\infty \|J\|_2 \;\text{ and }\; \|\tilde{K}\|_2 \leq \|b\|_\infty \|K\|_2.$$

Let $T : L^2(\Omega_1 \times \Omega_2, H) \times L^2(\Omega_2 \times \Omega_3, H) \to \mathcal{S}^2(H_1, H_3)$ be the bilinear map defined for all $F \in L^2(\Omega_1 \times \Omega_2, H)$ and $G \in L^2(\Omega_2 \times \Omega_3, H)$ and for almost every (t_1, t_3) by

$$[T(F, G)](t_1, t_3) = \int_{\Omega_2} \langle F(t_1, t_2), G(t_2, t_3)\rangle\, \mathrm{d}\mu_2(t_2).$$

We will show that T is actually valued in $\mathcal{S}^1(H_1, H_3)$ and that for all F and G as above we have

$$\|T(F, G)\|_1 \leq \|F\|_2 \|G\|_2.$$

By density, it is enough to prove this inequality when F and G have the form

$$F = \sum_{i=1}^{n} h_i \chi_{A_i \times B_i} \ \text{ and } \ G = \sum_{j=1}^{n} k_j \chi_{B_j \times C_j}$$

where for all i, j, $h_i, k_j \in H$ and $A_1, \ldots, A_n$ (respectively $B_1, \ldots, B_n$ and $C_1, \ldots, C_n$) are pairwise disjoint measurable subsets of Ω_1 (respectively Ω_2 and Ω_3) with finite measure. For such F and G we have

$$\|F\|_2 = \left(\sum_{i=1}^{n} \|h_i\|^2 \mu_1(A_i)\mu_2(B_i) \right)^{1/2} \quad \text{and} \|G\|_2 = \left(\sum_{j=1}^{n} \|k_j\|^2 \mu_2(B_j)\mu_3(C_j) \right)^{1/2}.$$

We have, for almost every (t_1, t_3),

$$\begin{aligned} T(F,G)(t_1,t_3) &= \sum_{i,j=1}^{n} \langle h_i, k_j \rangle \int_{\Omega_2} \chi_{A_i}(t_1)\chi_{B_i}(t_2)\chi_{B_j}(t_2)\chi_{C_j}(t_3) \mathrm{d}\mu_2(t_2) \\ &= \sum_{i=1}^{n} \langle h_i, k_i \rangle \int_{\Omega_2} \chi_{A_i}(t_1)\chi_{B_i}(t_2)\chi_{B_i}(t_2)\chi_{C_i}(t_3) \mathrm{d}\mu_2(t_2). \end{aligned}$$

Therefore, for all $h \in L^2(\Omega_1)$,

$$\begin{aligned} [T(F,G)](h) &= \sum_{i=1}^{n} \langle h_i, k_i \rangle \int_{\Omega_2} \left(\int_{\Omega_1} \chi_{A_i}(t_1)\chi_{B_i}(t_2) h(t_1) \mathrm{d}\mu_1(t_1) \right) \chi_{B_i}(t_2)\chi_{C_i}(\cdot) \mathrm{d}\mu_2(t_2) \\ &= \sum_{i=1}^{n} \langle h_i, k_i \rangle \left(X_{\chi_{A_i \times B_i}} \circ X_{\chi_{B_i \times C_i}} \right)(h) \end{aligned}$$

where for all i, $X_{\chi_{A_i \times B_i}} \in \mathcal{S}^2(H_1, H_2)$ and $X_{\chi_{B_i \times C_i}} \in \mathcal{S}^2(H_2, H_3)$ are defined in (1.15). Thus, by Cauchy-Schwarz inequality and the isometry (1.16),

$$\begin{aligned} \|T(F,G)\|_1 &\leq \sum_{i=1}^{n} | \langle h_i, k_i \rangle | \|X_{\chi_{A_i \times B_i}}\|_2 \|X_{\chi_{B_i \times C_i}}\|_2 \\ &\leq \sum_{i=1}^{n} \|h_i\| \|k_i\| \|\chi_{A_i \times B_i}\|_2 \|\chi_{B_i \times C_i}\|_2 \\ &\leq \left(\sum_{i=1}^{n} \|h_i\|^2 \mu_1(A_i)\mu_2(B_i) \right)^{1/2} \left(\sum_{j=1}^{n} \|k_j\|^2 \mu_2(B_j)\mu_3(C_j) \right)^{1/2} \\ &= \|F\|_2 \|G\|_2. \end{aligned}$$

We deduce that for all $F \in L^2(\Omega_1 \times \Omega_2, H)$ and $G \in L^2(\Omega_2 \times \Omega_3, H)$,

$$\|T(F,G)\|_1 \leq \|F\|_2 \|G\|_2.$$

Finally, for all $J \in \mathcal{S}^2(H_1, H_2)$ and $K \in \mathcal{S}^2(H_2, H_3)$ we have

$$\begin{aligned}\|\Lambda(\phi)(J,K)\|_1 &= \|F(\tilde{J},\tilde{K}\|_1 \\ &\leq \|\tilde{J}\|_2\|\tilde{K}\|_2 \\ &\leq \|a\|_\infty\|b\|_\infty\|J\|_2\|K\|_2\end{aligned}$$

This proves $(ii) \Rightarrow (i)$ with the estimate

$$\|\Lambda(\phi) : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\| \leq \|a\|_\infty\|b\|_\infty.$$

□

3.4 Perspectives

In Section 3.1, we defined bilinear Schur multipliers as mappings defined on $\mathcal{S}^2 \times \mathcal{S}^2$. As mentioned, we can also define bilinear Schur multiplier from $\mathcal{S}^p \times \mathcal{S}^q$ into $\mathcal{S}^r$, whenever $1 \leq p, q, r \leq \infty$. In this setting, it would be interesting to find a formula for the norm of a bilinear Schur multiplier from $\mathcal{S}^p \times \mathcal{S}^q$ into $\mathcal{S}^r$, similar to the one given in Theorem 3.4. Note that it is possible to see that when $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, a description of bilinear Schur multipliers would imply a description of linear Schur multipliers on $\mathcal{S}^q$. As said in Chapter 2, the only cases for which we have such description are $q = 1, q = 2$ and $q = \infty$. However, it is probably possible to find sufficient conditions for a bilinear Schur multiplier defined on $\mathcal{S}^p \times \mathcal{S}^q$ to be valued in $\mathcal{S}^1$, when p and q are conjugate exponents.

Of course, all the questions we can state concerning bilinear Schur multipliers can be stated as well in the continuous setting. We saw in Section 3.3 that the passage from the classical to the continuous case is not straightforward and required some additional studies such as the measurable factorization in $L^\infty_\sigma(\Omega; \Gamma_2(L^1, L^\infty))$ (see Section 1.4). However, one can try to prove that a result in the classical (or finite-dimensional) case implies, by approximation, a similar result in the continuous case.

Chapter 4

Multiple Operator Integrals

Let $\mathcal{H}$ be a separable Hilbert space and let A, B be two normal operators on $\mathcal{H}$. Any bounded Borel function ϕ on $\sigma(A) \times \sigma(B)$ gives rise to a double operator integral mapping $\Gamma^{A,B}(\phi)\colon \mathcal{S}^2(\mathcal{H}) \to \mathcal{S}^2(\mathcal{H})$ formally defined as

$$\Gamma^{A,B}(\phi)(X) = \int_{\sigma(A)\times\sigma(B)} \phi(s,t)\,\mathrm{d}E^A(s)\,X\,\mathrm{d}E^B(t), \qquad X \in \mathcal{S}^2(\mathcal{H}),$$

where E^A and E^B denote the spectral measures of A and B, respectively. The theory of double operator integrals was settled and developed in a series of papers of Birman-Solomiak [BS66; BS67; BS73] and plays a major role in various aspects of operator theory, especially in the perturbation theory. We refer the reader to the survey papers [BS03; Pel16] for a comprehensive presentation of this topic and its applications. See also Chapter 5 for some results about perturbation theory for selfadjoint and unitary operators.

In this chapter, we first define more generally multiple operator integrals as multilinear mappings defined on a product of copies of $\mathcal{S}^2(\mathcal{H})$ and valued in $\mathcal{S}^2(\mathcal{H})$. We will see in Section 4.2 that in the finite dimensional case, double and triple operator integrals behave like linear and bilinear Schur multipliers.

In [Pel85], V.V. Peller gave a characterization of double operator integral mappings which restrict to a bounded operator on $\mathcal{S}^1(\mathcal{H})$. He showed that $\Gamma^{A,B}(\phi)$ is a bounded operator from $\mathcal{S}^1(\mathcal{H})$ into itself if and only there exist a Hilbert space H and two functions $a \in L^\infty(E^A; H)$ and $b \in L^\infty(E^B; H)$ such that

$$\phi(s,t) = \langle a(s), b(t)\rangle \qquad a.e.\text{-}(s,t).$$

This property means that the operator $L^1(E^A) \to L^\infty(E^B)$ with kernel ϕ factors through a Hilbert space. We refer to [Pel85] and [HK03] for other equivalent formulations. In Section 4.3, we study an analogue of Peller's Theorem for triple operator integrals (see Theorem 4.10). This result is an operator version of Theorem 3.8. We will actually apply this result to prove Theorem 4.10. In order to do this we will show in Subsection 4.1.3 a connection between continuous bilinear Schur multipliers and triple operator integrals.

Finally, in Section 4.4, we will prove a characterization, similar to the one in [KJT09], concerning the complete boundedness of triple operator integrals from $\mathcal{S}^\infty(\mathcal{H})\overset{h}{\otimes}\mathcal{S}^\infty(\mathcal{H})$ into $\mathcal{S}^\infty(\mathcal{H})$. This section will use all the results that we recalled in Subsection 1.1.3 concerning the Haagerup tensor product.

4.1 Definition

Multiple operator integrals appeared in many recent papers with various definitions, see in particular [ANP15; ANP16; AP17; NAS09; Pel06; DPS13]. In this section we provide a definition of triple operator integrals associated to a triple (A, B, C) of normal operators on $\mathcal{H}$, based on the construction of a natural w^*-continuous mapping from $L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$ into $\mathcal{B}_2(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}), \mathcal{S}^2(\mathcal{H}))$, see Theorem 4.3. This mapping is actually an isometry. Further the construction extends to multiple operator integrals, see Proposition 4.4. It turns out that this construction is equivalent to an old definition of multiple operator integrals due to Pavlov [Pav69], this will be explained in Remark 4.5.

4.1.1 Normal operators and scalar-valued spectral measures

We assume that the reader is familiar with the general spectral theory of normal operators on Hilbert space, for which we refer e.g. to [Rud73, Chapters 12 and 13] and [Con00, Sections 14 and 15]. Let $\mathcal{H}$ be a separable Hilbert space and let A be a (possibly unbounded) normal operator on $\mathcal{H}$. We let $\sigma(A)$ denote the spectrum of A and we let E^A denote the spectral measure of A, defined on the Borel subsets of $\sigma(A)$.

By definition a scalar-valued spectral measure for A is a positive finite measure λ_A on the Borel subsets of $\sigma(A)$, such that λ_A and E^A have the same sets of measure zero. Such measures exist, thanks to the separability assumption on $\mathcal{H}$. Indeed let

$$W^*(A) \subset \mathcal{B}(\mathcal{H})$$

be the von Neumann algebra generated by the range of E^A, then $W^*(A)$ has a separating vector e and

$$\lambda_A := \|E^A(.)e\|^2 \tag{4.1}$$

is a scalar-valued spectral measure for A. See [Con00, Sections 14 and 15] for details ; the argument there is given for a bounded A but readily extends to the unbounded case.

The Borel functional calculus for A takes any bounded Borel function $f\colon \sigma(A) \to \mathbb{C}$ to the bounded operator

$$f(A) := \int_{\sigma(A)} f(t)\, \mathrm{d}E^A(t)\,.$$

According to [Con00, Theorem 15.10], it induces a w^*-continuous $*$-representation

$$\pi_A\colon L^\infty(\lambda_A) \longrightarrow \mathcal{B}(\mathcal{H}). \tag{4.2}$$

As a matter of fact, the space $L^\infty(\lambda_A)$ does not depend on the choice of the scalar-valued spectral measure λ_A.

4.1.2 Multiple operator integrals associated with operators

Let $\mathcal{H}$ be a separable Hilbert space and let A, B, C be (possibly unbounded) normal operators on $\mathcal{H}$. Denote by E^A, E^B and E^C their spectral measures and let λ_A, λ_B and λ_C be scalar-valued spectral measures for A, B and C (see Subsection 4.1.1).

Let $\mathcal{E}_1 \subset L^\infty(\lambda_A)$, $\mathcal{E}_2 \subset L^\infty(\lambda_B)$ and $\mathcal{E}_3 \subset L^\infty(\lambda_C)$ be the spaces of simple functions on $(\sigma(A), \lambda_A)$, $(\sigma(B), \lambda_B)$ and $(\sigma(C), \lambda_C)$, respectively. We let

$$\Gamma\colon \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3 \longrightarrow \mathcal{B}_2(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}), \mathcal{S}^2(\mathcal{H}))$$

be the unique linear map such that

$$\Gamma(f_1 \otimes f_2 \otimes f_3)(X, Y) = f_1(A)Xf_2(B)Yf_3(C) \tag{4.3}$$

for any $f_1 \in \mathcal{E}_1$, $f_2 \in \mathcal{E}_2$ and $f_3 \in \mathcal{E}_3$, and for any $X, Y \in \mathcal{S}^2(\mathcal{H})$.

Lemma 4.1. *For all $\phi \in \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3$, and for all $X, Y \in \mathcal{S}^2(\mathcal{H})$, we have*

$$\|\Gamma(\phi)(X, Y)\|_2 \leq \|\phi\|_\infty \|X\|_2 \|Y\|_2.$$

Proof. Let $\phi \in \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3$. There exists a finite family $(F_i^1)_i$ (respectively $(F_j^2)_j$ and $(F_k^3)_k$) of pairwise disjoint measurable subsets of $\sigma(A)$ (respectively of $\sigma(B)$ and $\sigma(C)$) of positive measures, as well as a family $(m_{ijk})_{i,j,k}$ of complex numbers such that

$$\phi = \sum_{i,j,k} m_{ijk}\, \chi_{F_i^1} \otimes \chi_{F_j^2} \otimes \chi_{F_k^3}. \tag{4.4}$$

Then we have

$$\|\phi\|_\infty = \sup_{i,j,k} |m_{ijk}|. \tag{4.5}$$

Let $X, Y \in \mathcal{S}^2(\mathcal{H})$. According to the definition of Γ, we have

$$\Gamma(\phi)(X, Y) = \sum_{i,j,k} m_{ijk} E^A(F_i^1) X E^B(F_j^2) Y E^C(F_k^3).$$

By the pairwise disjointnesses of $(F_i^1)_i$ and $(F_k^3)_k$, the elements

$$\Big(\sum_j m_{ijk} E^A(F_i^1) X E^B(F_j^2) Y E^C(F_k^3)\Big)_{i,k}$$

are pairwise orthogonal in $\mathcal{S}^2(\mathcal{H})$. Hence

$$\|\Gamma(\phi)(X, Y)\|_2^2 = \sum_{i,k} \Big\| \sum_j m_{ijk} E^A(F_i^1) X E^B(F_j^2) Y E^C(F_k^3) \Big\|_2^2.$$

Applying the Cauchy-Schwarz inequality and (4.5), we deduce that

$$\begin{aligned}\|\Gamma(\phi)(X,Y)\|_2^2 &\leq \|\phi\|_\infty^2 \sum_{i,k}\Big(\sum_j \big\|E^A(F_i^1)XE^B(F_j^2)\big\|_2 \big\|E^B(F_j^2)YE^C(F_k^3)\big\|_2\Big)^2 \\ &\leq \|\phi\|_\infty^2 \sum_{i,k}\Big(\sum_j \big\|E^A(F_i^1)XE^B(F_j^2)\big\|_2^2\Big)\Big(\sum_j \big\|E^B(F_j^2)YE^C(F_k^3)\big\|_2^2\Big) \\ &\leq \|\phi\|_\infty^2 \Big(\sum_{i,j} \big\|E^A(F_i^1)XE^B(F_j^2)\big\|_2^2\Big)\Big(\sum_{j,k} \big\|E^B(F_j^2)YE^C(F_k^3)\big\|_2^2\Big).\end{aligned}$$

Since the elements $E^A(F_i^1)XE^B(F_j^2)$ are pairwise orthogonal in $\mathcal{S}^2(\mathcal{H})$ we have

$$\begin{aligned}\sum_{i,j} \big\|E^A(F_i^1)XE^B(F_j^2)\big\|_2^2 &= \Big\|\sum_{i,j} E^A(F_i^1)XE^B(F_j^2)\Big\|_2^2 \\ &= \big\|E^A\left(\cup_i F_i^1\right) XE^B\left(\cup_j F_j^2\right)\big\|_2^2 \\ &\leq \|X\|_2^2.\end{aligned}$$

Similarly,

$$\sum_{j,k} \big\|E^B(F_j^2)YE^C(F_k^3)\big\|_2^2 \leq \|Y\|_2^2.$$

This yields the result. □

We let

$$G := \overline{\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3}^{\|\cdot\|_\infty} \subset L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$$

and we let $\tau\colon L^1(\lambda_A \times \lambda_B \times \lambda_C) \to G^*$ be the canonical map defined by

$$\langle \tau(\varphi), \phi \rangle = \int_{\sigma(A)\times\sigma(B)\times\sigma(C)} \varphi\phi \,\mathrm{d}(\lambda_A \times \lambda_B \times \lambda_C)\,, \qquad \varphi \in L^1,\ \phi \in G.$$

This is obviously a contraction.

We claim that τ is actually an isometry. To check this fact, consider $\varphi \in \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3$, that we write as a finite sum

$$\varphi = \sum_{i,j,k} c_{ijk}\, \chi_{F_i^1} \otimes \chi_{F_j^2} \otimes \chi_{F_k^3},$$

with $c_{ijk} \in \mathbb{C}^*$ and $(F_i^1)_i$ (respectively $(F_j^2)_j$ and $(F_k^3)_k$) being pairwise disjoint measurable subsets of $\sigma(A)$ (respectively of $\sigma(B)$ and $\sigma(C)$), with positive measures. Then

$$\|\varphi\|_1 = \sum_{i,j,k} |c_{ijk}|\, \lambda_A(F_i^1)\lambda_B(F_j^2)\lambda_C(F_k^3).$$

Let ϕ be defined by (4.4), with $m_{ijk} = |c_{ijk}|c_{ijk}^{-1}$. Then $\|\phi\|_\infty = 1$ by (4.5) and

$$\langle \tau(\varphi), \phi \rangle = \sum_{i,j,k} m_{ijk}c_{ijk}\lambda_A(F_i^1)\lambda_B(F_j^2)\lambda_C(F_k^3) = \|\varphi\|_1.$$

Hence we have $\|\tau(\varphi)\| = \|\varphi\|_1$ as expected. Since $\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3$ is dense in $L^1(\lambda_A \times \lambda_B \times \lambda_C)$, this implies that τ is an isometry. Thus we now consider $L^1(\lambda_A \times \lambda_B \times \lambda_C)$ as a subspace of G^*.

Arguing as in Subsection 3.3.1, we have isometric identifications

$$\begin{aligned}\mathcal{B}_2(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}), \mathcal{S}^2(\mathcal{H})) &= \mathcal{B}(\mathcal{S}^2(\mathcal{H}) \overset{\wedge}{\otimes} \mathcal{S}^2(\mathcal{H}), \mathcal{S}^2(\mathcal{H})) \\ &= \big(\mathcal{S}^2(\mathcal{H}) \overset{\wedge}{\otimes} \mathcal{S}^2(\mathcal{H}) \overset{\wedge}{\otimes} \mathcal{S}^2(\mathcal{H})\big)^*,\end{aligned}$$

and it is easy to check that the duality pairing providing this identification reads

$$\big\langle T, X \otimes Y \otimes Z\big\rangle = \mathrm{tr}\big(T(X,Y)Z\big)$$

for any $T \in \mathcal{B}_2(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}), \mathcal{S}^2(\mathcal{H}))$ and any $X, Y, Z \in \mathcal{S}^2(\mathcal{H})$.

We set

$$E := \mathcal{S}^2(\mathcal{H}) \overset{\wedge}{\otimes} \mathcal{S}^2(\mathcal{H}) \overset{\wedge}{\otimes} \mathcal{S}^2(\mathcal{H}).$$

According to Lemma 4.1, Γ uniquely extends to a contraction

$$\widetilde{\Gamma} \colon G \longrightarrow \mathcal{B}_2(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}), \mathcal{S}^2(\mathcal{H})) = E^*.$$

We can therefore consider $S = \widetilde{\Gamma}^*_{|E} : E \to G^*$, the restriction of $\widetilde{\Gamma}^*$ to $E \subset E^{**}$.

Lemma 4.2. *The operator S takes its values in the subspace $L^1(\lambda_A \times \lambda_B \times \lambda_C)$ of G^*.*

Proof. Let $\mathcal{P} = \overline{\mathcal{H}} \otimes \mathcal{H} \otimes \overline{\mathcal{H}} \otimes \mathcal{H} \otimes \overline{\mathcal{H}} \otimes \mathcal{H}$. Recall that we identify $\overline{\mathcal{H}} \otimes \mathcal{H}$ with the space of finite rank operators on $\mathcal{H}$. Then $\overline{\mathcal{H}} \otimes \mathcal{H}$ is a dense subspace of $\mathcal{S}^2(\mathcal{H})$. Consequently $\mathcal{P}$ is a dense subspace of E. Since S is continuous, it therefore suffices to show that $S(\mathcal{P}) \subset L^1(\lambda_A \times \lambda_B \times \lambda_C)$. Consider $\eta_1, \eta_2, \eta_3, \xi_1, \xi_2, \xi_3$ in $\mathcal{H}$ and $\omega = \overline{\xi_1} \otimes \eta_1 \otimes \overline{\xi_2} \otimes \eta_2 \otimes \overline{\xi_3} \otimes \eta_3$. Such elements span $\mathcal{P}$ hence it suffices to check that $S(\omega)$ belongs to $L^1(\lambda_A \times \lambda_B \times \lambda_C)$. Let $f_1 \in \mathcal{E}_1$, $f_2 \in \mathcal{E}_2$ and $f_3 \in \mathcal{E}_3$. We have

$$\begin{aligned}\langle S(\omega), f_1 \otimes f_2 \otimes f_3\rangle &= \langle \omega, \Gamma(f_1 \otimes f_2 \otimes f_3)\rangle \\ &= \mathrm{tr}\Big(\big[\Gamma(f_1 \otimes f_2 \otimes f_3)(\overline{\xi_1} \otimes \eta_1, \overline{\xi_2} \otimes \eta_2)\big](\overline{\xi_3} \otimes \eta_3)\Big) \\ &= \mathrm{tr}\big(f_1(A)(\overline{\xi_1} \otimes \eta_1) f_2(B)(\overline{\xi_2} \otimes \eta_2) f_3(C)(\overline{\xi_3} \otimes \eta_3)\big) \\ &= \mathrm{tr}\big((\overline{\xi_1} \otimes f_1(A)\eta_1)(\overline{\xi_2} \otimes f_2(B)\eta_2)(\overline{\xi_3} \otimes f_3(C)\eta_3)\big) \\ &= \mathrm{tr}\big((\overline{\xi_3} \otimes f_1(A)\eta_1)\,\langle f_3(C)\eta_3, \xi_2\rangle\,\langle f_2(B)\eta_2, \xi_1\rangle\big) \\ &= \langle f_3(C)\eta_3, \xi_2\rangle\,\langle f_2(B)\eta_2, \xi_1\rangle\,\langle f_1(A)\eta_1, \xi_3\rangle\,.\end{aligned}$$

The w^*-continuity of the functional calculus $*$-representation $\pi_A \colon L^\infty(\lambda_A) \to B(\mathcal{H})$ to which we refer in (4.2) tells us that

$$\langle f_1(A)\eta_1, \xi_3\rangle = \int_{\sigma(A)} f_1 h_1 \,\mathrm{d}\lambda_A$$

for some $h_1 \in L^1(\lambda_A)$ not depending on f_1. A thorough look at the construction of π_A shows that h_1 is actually the Radon-Nikodym derivative of the measure $dE^A_{\eta_1,\xi_3}$ with respect to λ_A.

Similarly, there exist $h_2 \in L^1(\lambda_B)$ and $h_3 \in L^1(\lambda_C)$ not depending on f_2 and f_3 such that $\langle f_2(B)\eta_2, \xi_1\rangle = \int_{\sigma(B)} f_2 h_2 \, \mathrm{d}\lambda_B$ and $\langle f_3(C)\eta_3, \xi_2\rangle = \int_{\sigma(C)} f_3 h_3 \, \mathrm{d}\lambda_C$. Consequently,

$$\langle S(\omega), f_1 \otimes f_2 \otimes f_3\rangle = \int_{\sigma(A)\times\sigma(B)\times\sigma(C)} (f_1 \otimes f_2 \otimes f_3)(h_1 \otimes h_2 \otimes h_3) \, \mathrm{d}(\lambda_A \times \lambda_B \times \lambda_C).$$

Since $\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3$ is dense in G, this implies that

$$S(\omega) = h_1 \otimes h_2 \otimes h_3 \in L^1(\lambda_A \times \lambda_B \times \lambda_C).$$

□

Theorem 4.3. *There exists a unique w^*-continuous isometry*

$$\Gamma^{A,B,C} \colon L^\infty(\lambda_A \times \lambda_B \times \lambda_C) \longrightarrow \mathcal{B}_2(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}), \mathcal{S}^2(\mathcal{H})),$$

such that for any $f_1 \in \mathcal{E}_1$, $f_2 \in \mathcal{E}_2$ and $f_3 \in \mathcal{E}_3$, and for any $X, Y \in \mathcal{S}^2(\mathcal{H})$, we have

$$\Gamma^{A,B,C}(f_1 \otimes f_2 \otimes f_3)(X, Y) = f_1(A) X f_2(B) Y f_3(C).$$

Proof. The uniqueness follows from the w^*-density of $\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3$ in $L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$.

Lemma 4.2 yields $S \colon E \to L^1(\lambda_A \times \lambda_B \times \lambda_C)$. Then its adjoint $\mathcal{S}^*$ is a w^*-continuous contraction from $L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$ into $E^* = \mathcal{B}_2(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}), \mathcal{S}^2(\mathcal{H}))$. We set

$$\Gamma^{A,B,C} = S^*.$$

By construction, $\Gamma^{A,B,C}$ is w^*-continuous and extends the map Γ defined by (4.3). The fact that $\Gamma^{A,B,C}$ is an isometry will be proved later on in Corollary 4.8. □

Bilinear maps of the form $\Gamma^{A,B,C}(\phi)$ will be called *triple operator integral mappings* in this thesis. Operators of the form $\Gamma^{A,B,C}(\phi)(X,Y) \colon \mathcal{H} \to \mathcal{H}$ are called triple operator integrals.

By similar computations (left to the reader), the above construction can be extended to $(n-1)$-tuple operator integrals, for any $n \geq 2$. One obtains the following statement, in which $\mathcal{B}_{n-1}(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}) \times \cdots \times \mathcal{S}^2(\mathcal{H}), \mathcal{S}^2(\mathcal{H}))$ denotes the space of $(n-1)$-linear bounded maps from the product of $(n-1)$ copies of $\mathcal{S}^2(\mathcal{H})$ taking values in $\mathcal{S}^2(\mathcal{H})$.

Proposition 4.4. *Let $n \geq 2$ and let $A_1, A_2, \ldots, A_n$ be normal operators on $\mathcal{H}$. For any $i = 1, \ldots, n$, let λ_{A_i} be a scalar-valued spectral measure for A_i and let $\mathcal{E}_i \subset L^\infty(\lambda_{A_i})$ be the space of simple functions on $(\sigma(A_i), \lambda_{A_i})$. There exists a unique w^*-continuous linear isometry*

$$\Gamma^{A_1,A_2,\ldots,A_n} : L^\infty\left(\prod_{i=1}^n \lambda_{A_i}\right) \longrightarrow \mathcal{B}_{n-1}(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}) \times \cdots \times \mathcal{S}^2(\mathcal{H}) \to \mathcal{S}^2(\mathcal{H})),$$

such that for any $f_i \in \mathcal{E}_i$ and for any $X_1, \ldots, X_{n-1} \in \mathcal{S}^2(\mathcal{H})$, we have

$$\Gamma^{A_1,A_2,\ldots,A_n}(f_1 \otimes \cdots \otimes f_n)(X_1, \ldots, X_{n-1}) = f_1(A_1) X_1 f_2(A_2) \cdots f_{n-1}(A_{n-1}) X_{n-1} f_n(A_n).$$

Remark 4.5. *As indicated in the introduction of this section, the above construction turns out to be equivalent to Pavlov's definition of multiple operator integrals given in [Pav69]. Let us briefly review Pavlov's construction from [Pav69], and explain this 'equivalence'. In this remark, we use terminology and references from [DU79, Chapter 1].*

Let $n \geq 2$ and consider normal operators $A_1, A_2, \ldots, A_n$ as in Proposition 4.4. Fix $X_1, \ldots, X_{n-1}$ in $\mathcal{S}^2(\mathcal{H})$. Let $\Omega := \sigma(A) \times \sigma(A_2) \times \cdots \times \sigma(A_n)$ and consider the set $\mathcal{F}$ consisting of finite unions of subsets of Ω of the form

$$\Delta = F_1 \times F_2 \times \cdots \times F_n,$$

where, for any $1 \leq i \leq n$, F_i is a Borel subset of $\sigma(A_i)$.

There exists a (necessarily unique) finitely additive vector measure $m \colon \mathcal{F} \to \mathcal{S}^2(\mathcal{H})$ such that

$$m(\Delta) = E^{A_1}(F_1)X_1E^{A_2}(F_2)\cdots E^{A_{n-1}}(F_{n-1})X_{n-1}E^{A_n}(F_n) \tag{4.6}$$

for any Δ as above.

Pavlov first shows that m is a measure of bounded semivariation and then proves that m is actually countably additive (see [Pav69, Theorem 1]). Let $\mathcal{T}$ be the σ-field generated by $\mathcal{F}$. Since $\mathcal{S}^2(\mathcal{H})$ is reflexive, it follows from [DU79, Chapter 1, Section 5, Theorem 2] that m has a (necessarily unique) countably additive extension $\widetilde{m} \colon \mathcal{T} \to \mathcal{S}^2(\mathcal{H})$. Moreover $\widetilde{m}$ is a measure of bounded semivariation. Then using the fact that for all i, λ_{A_i} is a scalar-valued spectral measure for A_i, one can show that

$$\widetilde{m} \ll \lambda_{A_1} \times \lambda_{A_2} \times \cdots \times \lambda_{A_n}$$

on $\mathcal{F}$. This implies that $L^\infty(\lambda_{A_1} \times \lambda_{A_2} \times \cdots \times \lambda_{A_n}) \subset L^\infty(\widetilde{m})$ and hence, for any $\phi \in L^\infty(\lambda_{A_1} \times \lambda_{A_2} \times \cdots \times \lambda_{A_n})$, one may define an integral

$$\int_\Omega \phi(t)\, d\widetilde{m}(t) \ \in \mathcal{S}^2(\mathcal{H}).$$

See [DU79, Chapter 1, Section 1, Theorem 13] for details. This element is defined in [Pav69] as the multiple operator integral associated to ϕ and $(X_1, \ldots, X_{n-1})$.

We claim that this construction is equivalent to the one given in the present thesis, namely

$$\int_\Omega \phi(t)\, d\widetilde{m}(t) = \Gamma^{A_1,A_2,\ldots,A_n}(\phi)(X_1, \ldots, X_{n-1}).$$

To check this identity, let $w_1, w_2 \colon L^\infty(\lambda_{A_1} \times \lambda_{A_2} \times \cdots \times \lambda_{A_n}) \to \mathcal{S}^2(\mathcal{H})$ be defined by $w_1(\phi) = \int_\Omega \phi(t)\, d\widetilde{m}(t)$ and $w_2(\phi) = \Gamma^{A_1,A_2,\ldots,A_n}(\phi)(X_1, \ldots, X_{n-1})$. For any $Z \in \mathcal{S}^2(\mathcal{H})$, the functional of $L^\infty(\lambda_{A_1} \times \lambda_{A_2} \times \cdots \times \lambda_{A_n})$ taking ϕ to $\left\langle \int_\Omega \phi(t)\, d\widetilde{m}(t)\, , Z \right\rangle$ induces a countably additive measure on $\mathcal{T}$, which is absolutely continuous with respect to $\lambda_{A_1} \times \lambda_{A_2} \times \cdots \times \lambda_{A_n}$. By the Radon-Nikodym Theorem, this functional is therefore w^*-continuous. This implies that w_1 is w^*-continuous. We know that w_2 is w^*-continuous as well, by Proposition 4.4. Further it is easy to derive from (4.6) that w_1 and w_2 coincide on $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_n$. These properties imply the equality $w_1 = w_2$ as claimed.

Remark 4.6. *We keep the notations from Proposition 4.4 and explain the connection of this result with Peller's construction from [Pel06]. In the latter paper, the author defines multiple* *operator integrals associated to functions belonging to the so-called integral projective tensor product of the spaces $L^\infty(\lambda_{A_i})$. We will check that this definition is consistent with ours.*

Let $(\Sigma, d\mu)$ be a σ-finite measure space and, for any $i = 1, \dots, n$, let $a_i \colon \Sigma \times \sigma(A_i) \to \mathbb{C}$ be a measurable function such that $a_i(t, \cdot\,) \in L^\infty(\lambda_{A_i})$ for a.-e. $t \in \Sigma$. Assume that

$$\int_\Sigma \|a_1(t,\cdot\,)\|_{L^\infty(\lambda_{A_1})} \|a_2(t,\cdot\,)\|_{L^\infty(\lambda_{A_2})} \cdots \|a_n(t,\cdot\,)\|_{L^\infty(\lambda_{A_n})}\, d\mu(t) \,<\infty. \tag{4.7}$$

Then one may define $\phi \in L^\infty(\lambda_{A_1} \times \cdots \times \lambda_{A_n})$ by setting

$$\phi(t_1, t_2, \dots, t_n) = \int_\Sigma a_1(t, t_1) a_2(t, t_2) \cdots a_n(t, t_n)\, d\mu(t) \tag{4.8}$$

for a.-e. $(t_1, \dots, t_n)$ in $\sigma(A_1) \times \cdots \times \sigma(A_n)$. We claim that for any $X_1, \dots, X_{n-1} \in \mathcal{S}^2(\mathcal{H})$,

$$\Gamma^{A_1,\dots,A_n}(X_1, \dots, X_{n-1}) = \int_\Sigma a_1(t, A_1) X_1 a_2(t, A_2) X_2 \cdots X_{n-1} a_n(t, A_n)\, d\mu(t)\,, \tag{4.9}$$

where $a_i(t, A_i) \in B(\mathcal{H})$ is obtained by applying the Borel functional calculus of A_i to $a_i(t,\cdot\,)$, for any $i = 1, \dots, n$. The right-hand side of (4.9) is Peller's definition of the multiple operator integral associated with ϕ and $(X_1, \dots, X_{n-1})$. Hence the equality (4.9) shows that Peller's definition is a special case of Proposition 4.4. The reason why [Pel06] focuses on functions ϕ as above is that the right-hand side of (4.9) converges in $B(\mathcal{H})$ whenever $X_1, \dots, X_{n-1} \in B(\mathcal{H})$. Consequently, $\Gamma^{A_1,\dots,A_n}(\phi)$ extends to a bounded $n-1$-linear map $B(\mathcal{H}) \times \cdots \times B(\mathcal{H}) \to B(\mathcal{H})$ under the assumptions (4.7) and (4.8).

To prove (4.9), we introduce $\widetilde{a_i} \colon \Sigma \to L^\infty(\lambda_{A_i})$ by writing $\widetilde{a_i}(t) = a_i(t,\cdot)$, for any $i = 1, \dots, n$. Then the function $\widetilde{\phi} \colon \Sigma \to L^\infty(\lambda_{A_1} \times \cdots \times \lambda_{A_n})$ defined by

$$\widetilde{\phi}(t) = \widetilde{a_1}(t) \otimes \widetilde{a_2}(t) \otimes \cdots \otimes \widetilde{a_n}(t), \qquad t \in \Sigma,$$

is w^*-measurable and the associated norm function $\|\widetilde{\phi}(\cdot\,)\|_\infty$ is integrable, by the assumption (4.7). We can therefore consider its integral $\int_\Sigma \widetilde{\phi}(t)\, d\mu(t)$ as an element of $L^\infty(\lambda_{A_1} \times \cdots \times \lambda_{A_n})$, defined in the w^*-sense. Using Fubini's Theorem, one obtains that

$$\phi \,=\, \int_\Sigma \widetilde{\phi}(t)\, d\mu(t)\,,$$

where the function ϕ is defined by (4.8). Since $\Gamma^{A_1,\dots,A_n}$ is w^*-continuous, we derive that

$$\Gamma^{A_1,\dots,A_n}(\phi) \,=\, \int_\Sigma \Gamma^{A_1,\dots,A_n}\big(\widetilde{a_1}(t) \otimes \widetilde{a_2}(t) \otimes \cdots \otimes \widetilde{a_n}(t)\big)\, d\mu(t).$$

Let $X_1, \dots, X_{n-1} \in \mathcal{S}^2(\mathcal{H})$. We deduce that

$$\Gamma^{A_1,\dots,A_n}(\phi)(X_1, \dots, X_{n-1}) = \int_\Sigma \Gamma^{A_1,\dots,A_n}\big(\widetilde{a_1}(t) \otimes \widetilde{a_2}(t) \otimes \cdots \otimes \widetilde{a_n}(t)\big)(X_1, \dots, X_{n-1})\, d\mu(t)$$

as a Bochner integral in $\mathcal{S}^2(\mathcal{H})$. The equality (4.9) now follows from the definition of $\Gamma^{A_1,\dots,A_n}$ on elementary tensor products.

4.1.3 Passing from operators to functions

Let $\mathcal{H}$ be a separable Hilbert space and let A, B and C be normal operators on $\mathcal{H}$. We keep the notations from Subsection 4.1.2. We associate the three measure spaces

$$(\Omega_1, \mu_1) = (\sigma(C), \lambda_C), \qquad (\Omega_2, \mu_2) = (\sigma(B), \lambda_B) \qquad \text{and} \qquad (\Omega_3, \mu_3) = (\sigma(A), \lambda_A)$$

and consider the mapping Λ defined in Subsection 3.3.1 for these three measure spaces. It maps $L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$ into $\mathcal{B}_2(\mathcal{S}^2(L^2(\lambda_C), L^2(\lambda_B)) \times \mathcal{S}^2(L^2(\lambda_B), L^2(\lambda_A)), \mathcal{S}^2(L^2(\lambda_C), L^2(\lambda_A)))$. The main purpose of this subsection is to establish a precise connection between this mapping Λ and the triple operator integral mapping $\Gamma^{A,B,C}$ from Theorem 4.3.

We may suppose that

$$\lambda_A(.) = \|E^A(.)e_1\|^2, \quad \lambda_B(.) = \|E^B(.)e_2\|^2 \quad \text{and} \quad \lambda_C(.) = \|E^C(.)e_3\|^2$$

for some vectors $e_1, e_2, e_3 \in \mathcal{H}$ (see Subsection 4.1.1).

There exists a (necessarily unique) linear map $\rho_A \colon \mathcal{E}_1 \longrightarrow \mathcal{H}$ satisfying

$$\rho_A(\chi_F) = E^A(F)e_1$$

for any Borel set $F \subset \sigma(A)$. For any finite family $(F_i)_i$ of pairwise disjoint measurable subsets of $\sigma(A)$ and for any family $(\alpha_i)_i$ of complex numbers, we have

$$\begin{aligned}
\Big\|\rho_A\Big(\sum_i \alpha_i \chi_{F_i}\Big)\Big\|^2 &= \Big\|\sum_i \alpha_i E^A(F_i)e_1\Big\|^2 \\
&= \sum_i |\alpha_i|^2 \|E^A(F_i)e_1\|^2 \\
&= \sum_i |\alpha_i|^2 \lambda_A(F_i) \\
&= \Big\|\sum_i \alpha_i \chi_{F_i}\Big\|_2^2.
\end{aligned}$$

Hence ρ_A extends to an isometry (still denoted by)

$$\rho_A : L^2(\lambda_A) \longrightarrow \mathcal{H}.$$

Denote by $\mathcal{H}_A$ the range of ρ_A. We obtain

$$L^2(\lambda_A) \overset{\rho_A}{\equiv} \mathcal{H}_{\mathcal{A}}.$$

Similarly, we define ρ_B, ρ_C and $\mathcal{H}_B, \mathcal{H}_C \subset \mathcal{H}$ such that

$$L^2(\lambda_B) \overset{\rho_B}{\equiv} \mathcal{H}_{\mathcal{B}} \quad \text{and} \quad L^2(\lambda_C) \overset{\rho_C}{\equiv} \mathcal{H}_{\mathcal{C}}.$$

We may regard $\mathcal{S}^2(\mathcal{H}_B, \mathcal{H}_A)$, $\mathcal{S}^2(\mathcal{H}_C, \mathcal{H}_B)$ and $\mathcal{S}^2(\mathcal{H}_C, \mathcal{H}_A)$ as subspaces of $\mathcal{S}^2(\mathcal{H})$ in a natural way, see (1.17). The next statement means that for any $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$, $\Gamma^{A,B,C}(\phi)$ maps $\mathcal{S}^2(\mathcal{H}_B, \mathcal{H}_A) \times \mathcal{S}^2(\mathcal{H}_C, \mathcal{H}_B)$ into $\mathcal{S}^2(\mathcal{H}_C, \mathcal{H}_A)$ and that under the previous identifications, this restriction 'coincides' with $\Lambda(\phi)$.

Proposition 4.7. *Let* $X \in \mathcal{S}^2(L^2(\lambda_B), L^2(\lambda_A))$ *and* $Y \in \mathcal{S}^2(L^2(\lambda_C), L^2(\lambda_B))$, *and set*

$$\widetilde{X} = \rho_A \circ X \circ \rho_B^{-1} \in \mathcal{S}^2(\mathcal{H}_B, \mathcal{H}_A) \qquad \text{and} \qquad \widetilde{Y} = \rho_B \circ Y \circ \rho_C^{-1} \in \mathcal{S}^2(\mathcal{H}_C, \mathcal{H}_B).$$

For any $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$, $\Gamma^{A,B,C}(\phi)(\widetilde{X}, \widetilde{Y})$ *belongs to* $\mathcal{S}^2(\mathcal{H}_C, \mathcal{H}_A)$ *and*

$$\Lambda(\phi)(Y, X) = \rho_A^{-1} \circ \Gamma^{A,B,C}(\phi)(\widetilde{X}, \widetilde{Y}) \circ \rho_C. \tag{4.10}$$

Proof. We first consider the special case when $\phi = \chi_{F_1} \otimes \chi_{F_2} \otimes \chi_{F_3}$ for some measurable subsets $F_1 \subset \sigma(A)$, $F_2 \subset \sigma(B)$ and $F_3 \subset \sigma(C)$.

Let $U \subset \sigma(A)$, $V, V' \subset \sigma(B)$ and $W \subset \sigma(C)$ and consider the elementary tensors

$$X = \chi_V \otimes \chi_U \in \mathcal{S}^2(L^2(\lambda_B), L^2(\lambda_A)) \quad \text{and} \quad Y = \chi_W \otimes \chi_{V'} \in \mathcal{S}^2(L^2(\lambda_C), L^2(\lambda_B)).$$

We associate $\widetilde{X}$ and $\widetilde{Y}$ as in the statement. Since $\rho_B \colon L^2(\lambda_B) \to \mathcal{H}_B$ is unitary, we have $\rho_B^{-1} = \rho_B^*$ hence

$$\widetilde{X} = \rho_B(\chi_V) \otimes \rho_A(\chi_U) = E^B(V)e_2 \otimes E^A(U)e_1.$$

Likewise,

$$\widetilde{Y} = E^C(W)e_3 \otimes E^B(V')e_2.$$

We have

$$\begin{aligned}
\Lambda(\phi)(Y, X) &= \int_{\sigma(B)} \phi(., t_2, .) X(t_2, .) Y(., t_2)\, \mathrm{d}\lambda_B(t_2) \\
&= \int_{\sigma(B)} \chi_{F_2}(t_2) \chi_V(t_2) \chi_{V'}(t_2)\, \chi_{F_3}\chi_W \otimes \chi_{F_1}\chi_U \, \mathrm{d}\lambda_B(t_2) \\
&= \left(\int_{F_2 \cap V \cap V'} \mathrm{d}\lambda_B(t_2) \right) \chi_{F_3 \cap W} \otimes \chi_{F_1 \cap U} \\
&= \lambda_B(F_2 \cap V \cap V')\, \chi_{F_3 \cap W} \otimes \chi_{F_1 \cap U}.
\end{aligned}$$

Further using the above expressions of $\widetilde{X}$ and $\widetilde{Y}$, we have

$$\begin{aligned}
\Gamma^{A,B,C}(\phi)(\widetilde{X}, \widetilde{Y}) &= E^A(F_1) \widetilde{X} E^B(F_2) \widetilde{Y} E^C(F_3) \\
&= \big(E^B(V)e_2 \otimes E^A(F_1 \cap U)e_1\big)\big(E^C(F_3 \cap W)e_3 \otimes E^B(F_2 \cap V')e_2\big) \\
&= \big\langle E^B(F_2 \cap V')e_2, E^B(V)e_2 \big\rangle\, E^C(F_3 \cap W)e_3 \otimes E^A(F_1 \cap U)e_1 \\
&= \big\langle E^B(F_2 \cap V' \cap V)e_2, e_2 \big\rangle\, E^C(F_3 \cap W)e_3 \otimes E^A(F_1 \cap U)e_1 \\
&= \lambda_B(F_2 \cap V \cap V')\, E^C(F_3 \cap W)e_3 \otimes E^A(F_1 \cap U)e_1.
\end{aligned}$$

This shows that $\Gamma^{A,B,C}(\phi)(\widetilde{X}, \widetilde{Y})$ belongs to $\mathcal{S}^2(\mathcal{H}_C, \mathcal{H}_A)$ and that (4.10) holds true.

By linearity and continuity, this result holds as well for all $X \in \mathcal{S}^2(L^2(\lambda_B), L^2(\lambda_A))$ and all $Y \in \mathcal{S}^2(L^2(\lambda_C), L^2(\lambda_B))$.

Finally since Λ and $\Gamma^{A,B,C}$ are w^*-continuous, we deduce from the above special case that the result actually holds true for all $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$. □

Corollary 4.8. *The mapping* $\Gamma^{A,B,C}$ *from Theorem 4.3 is an isometry.*

Proof. Consider $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$. For any X in $\mathcal{S}^2(L^2(\lambda_B), L^2(\lambda_A))$ and any Y in $\mathcal{S}^2(L^2(\lambda_C), L^2(\lambda_B))$, we have

$$\begin{aligned}\|\Lambda(\phi)(Y,X)\|_2 &= \|\rho_A^{-1} \circ \Gamma^{A,B,C}(\phi)(\widetilde{X},\widetilde{Y}) \circ \rho_C\|_2 \\ &\leq \|\Gamma^{A,B,C}(\phi)(\widetilde{X},\widetilde{Y})\|_2 \\ &\leq \big\|\Gamma^{A,B,C}(\phi)\big\| \|\widetilde{X}\|_2 \|\widetilde{X}\|_2\end{aligned}$$

by Proposition 4.7. Since $\|\widetilde{X}\|_2 = \|X\|_2$ and $\|\widetilde{Y}\|_2 = \|Y\|_2$, this implies that

$$\big\|\Lambda(\phi)\big\| \leq \big\|\Gamma^{A,B,C}(\phi)\big\|. \tag{4.11}$$

By Proposition 3.7, the left-hand side of this inequality is equal to $\|\phi\|_\infty$. Further $\Gamma^{A,B,C}$ is a contraction. Hence we obtain that $\|\Gamma^{A,B,C}(\phi)\| = \|\phi\|_\infty$. □

4.2 Finite dimensional case

In the previous section, we defined multiple operator integrals and we saw that we have a simple expression in the case when ϕ is in the tensor product of L^∞-spaces. Except for this case, we cannot give such a simple formula for any element of $L^\infty(\prod_{i=1}^n \lambda_{A_i})$. However, when the Hilbert space $\mathcal{H}$ is finite dimensional, it is possible to give a satisfying expression of multiple operator integrals : this is due to the fact that in this situation, we have a formula for functional calulus for selfadjoint operators, involving the eigenvalues and projections onto the eigenspaces. As a consequence, we will see that double and triple operator integrals behave like linear and bilinear Schur multipliers. It is straightforward to extend the formula we obtain here for multiple operator integrals. We use the results of this section in Chapter 5 to obtain norm estimates for multiple operator integrals in the finite dimensional case.

Throughout this section we work with finite-dimensional operators. We fix an integer $n \geq 1$ and regard $\mathbb{C}^n$ as equipped with its standard Hermitian structure.

Consider two orthonormal bases $e = \{e_j\}_{j=1}^n$ and $e' = \{e'_i\}_{i=1}^n$ in $\mathbb{C}^n$. Then every linear operator $A \in \mathcal{B}(\mathbb{C}^n)$ is associated with a matrix $A = \{a_{ij}\}_{i,j=1}^n$, where $a_{ij} = \langle A(e_j), e'_i\rangle$. Sometimes we use the notation $a_{ij}^{e',e}$ instead of a_{ij} to emphasize corresponding bases.

For any unit vector $x \in \mathbb{C}^n$ we let P_x denote the projection on the linear span of x, that is, $P_x(y) = \langle y, x\rangle x$ for any $y \in \mathbb{C}^n$.

4.2.1 Double operator integrals

Let $A, B \in \mathcal{B}(\mathbb{C}^n)$ be normal operators. Let $\xi_1 = \{\xi_i^{(1)}\}_{i=1}^n$ and $\xi_2 = \{\xi_i^{(2)}\}_{i=1}^n$ be orthonormal bases of eigenvectors for A and B respectively, and let $\{\lambda_i^{(j)}\}_{i=1}^n, j = 1, 2$ be the associated n-tuples of eigenvalues, that is, $A(\xi_i^{(1)}) = \lambda_i^{(1)}\xi_i^{(1)}$ and $B(\xi_i^{(2)}) = \lambda_i^{(2)}\xi_i^{(2)}$. Without loss of generality, we assume that $\{\lambda_i^{(j)}\}_{i=1}^{n_j}, j = 1, 2$, are the sets of pairwise distinct eigenvalues of A and B, where $n_j \in \mathbb{N}$, $n_j \leq n$. Denote

$$E_i^{(1)} = \sum_{\substack{k=1 \\ \lambda_k^{(1)}=\lambda_i^{(1)}}}^{n} P_{\xi_k^{(1)}}, \quad 1 \leq i \leq n_1, \tag{4.12}$$

that is, $E_i^{(1)}$ is a spectral projection of the operator A associated with the eigenvalue $\lambda_i^{(1)}$. Similary, we denote by $E_i^{(2)}$ a spectral projection of the operator B associated with the eigenvalue $\lambda_i^{(2)}$.

With those notations, we have

$$A = \sum_{i=1}^{n_1} \lambda_i^{(1)} E_i^{(1)} \quad \text{and} \quad B = \sum_{k=1}^{n_2} \lambda_k^{(2)} E_k^{(2)}. \tag{4.13}$$

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be a function. Then, the double operator integral $\Gamma^{A,B}(\phi) : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n)$ associated with ϕ, A and B defined in Proposition 4.4 is given by

$$\left[\Gamma^{A,B}(\phi)\right](X) = \sum_{i,k=1}^{n} \phi(\lambda_i^{(1)}, \lambda_k^{(2)}) P_{\xi_i^{(1)}} X P_{\xi_k^{(2)}}, \quad X \in \mathcal{B}(\mathbb{C}^n). \tag{4.14}$$

Alternatively, and it is sometimes more convenient, we can use the representation of $\left[\Gamma^{A,B}(\phi)\right](X)$ in the form

$$\left[\Gamma^{A,B}(\phi)\right](X) = \sum_{i=1}^{n_1} \sum_{k=1}^{n_2} \phi(\lambda_i^{(1)}, \lambda_k^{(2)}) E_i^{(1)} X E_k^{(2)}, \quad X \in \mathcal{B}(\mathbb{C}^n). \tag{4.15}$$

Let us prove Formula (4.15). Note that according to Proposition 4.4, we only need to know ϕ on $\sigma(A) \times \sigma(B)$. Let $F = \phi_{|\sigma(A)\times\sigma(B)}$. Then

$$F = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi(\lambda_i^{(1)}, \lambda_k^{(2)}) \chi_{\left\{\lambda_i^{(1)}\right\}} \otimes \chi_{\left\{\lambda_k^{(2)}\right\}}.$$

According to (4.13) we have, for $1 \leq i \leq n_1$,

$$\chi_{\left\{\lambda_i^{(1)}\right\}}(A) = \sum_{j=1}^{n_1} \chi_{\left\{\lambda_i^{(1)}\right\}}(\lambda_j^{(1)}) E_j^{(1)} = E_i^{(1)}.$$

Similarly, for any $1 \leq k \leq n_2$,

$$\chi_{\left\{\lambda_k^{(2)}\right\}}(B) = E_k^{(2)}.$$

Thus, for any $X \in \mathcal{B}(\mathbb{C}^n)$,

$$\left[\Gamma^{A,B}(\phi)\right](X) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi(\lambda_i^{(1)}, \lambda_k^{(2)}) \left[\Gamma^{A,B}\left(\chi_{\left\{\lambda_i^{(1)}\right\}} \otimes \chi_{\left\{\lambda_k^{(2)}\right\}}\right)\right](X)$$

$$= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \phi(\lambda_i^{(1)}, \lambda_k^{(2)}) \chi_{\{\lambda_i^{(1)}\}}(A) X \chi_{\{\lambda_k^{(2)}\}}(B)$$
$$= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \phi(\lambda_i^{(1)}, \lambda_k^{(2)}) E_i^{(1)} X E_k^{(2)}.$$

It is not difficult to see that if we identify $\mathcal{B}(\mathbb{C}^n)$ with M_n by associating X with the matrix $\{x_{ik}^{\xi_1,\xi_2}\}_{i,k=1}^n$, then the operator $\Gamma^{A,B}(\phi)$ acts as a linear Schur multiplier associated with $\{\phi(\lambda_i^{(1)}, \lambda_k^{(2)})\}_{i,k=1}^n$. Indeed,

$$\left\langle (P_{\xi_i^{(1)}} X P_{\xi_k^{(2)}})(\xi_s^{(2)}), \xi_r^{(1)} \right\rangle = \begin{cases} \langle X(\xi_s^{(2)}), \xi_r^{(1)} \rangle = x_{rs}^{\xi_1,\xi_2}, & \text{if } s = k,\ r = i, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\left\langle \left[\Gamma^{A,B}(\phi)\right](X)(\xi_k^{(2)}), \xi_i^{(1)} \right\rangle = \phi(\lambda_i^{(1)}, \lambda_k^{(2)}) x_{ik}^{\xi_1,\xi_2},$$

which implies that $\Gamma^{A,B}(\phi) \sim \left\{\phi(\lambda_i^{(1)}, \lambda_k^{(2)})\right\}_{i,k=1}^n : M_n \to M_n$. Since these identifications are isometric ones, we deduce that

$$\left\|\Gamma^{A,B}(\phi) \colon \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty\right\| = \left\|\{\phi(\lambda_i^{(1)}, \lambda_k^{(2)})\}_{i,k=1}^n \colon \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty\right\|. \tag{4.16}$$

4.2.2 Triple operator integrals

We now give the formula for triple operator integrals in the finite dimensional case.

Let $A, B, C \in \mathcal{B}(\mathbb{C}^n)$ be normal operators. We keep the same notations for the spectral decompositions of A and B introduced in the previous subsection. Let $\xi_3 = \{\xi_i^{(3)}\}_{i=1}^n$ be an orthornomal basis of eigenvectors of C and let $\{\lambda_i^{(3)}\}_{i=1}^n$ be the corresponding n-tuple of eigenvalues.

Let $\psi : \mathbb{C}^3 \to \mathbb{C}$ be a function. Then, the triple operator integral $\Gamma^{A,B,C}(\psi) : \mathcal{B}(\mathbb{C}^n) \times \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n)$ associated with ψ, A, B and C defined in Theorem 4.3 is given by

$$\left[\Gamma^{A,B,C}(\psi)\right](X, Y) = \sum_{i,j,k=1}^{n} \psi(\lambda_i^{(1)}, \lambda_k^{(2)}, \lambda_j^{(3)}) P_{\xi_i^{(1)}} X P_{\xi_k^{(2)}} Y P_{\xi_j^{(3)}} \tag{4.17}$$

for any $X, Y \in \mathcal{B}(\mathbb{C}^n)$.

Assume that $\{\lambda_i^{(3)}\}_{i=1}^{n_3}$ is the set of pairwise distinct eigenvalues of the operator C. Then alternatively, using the spectral projections (4.12), we can write

$$\left[\Gamma^{A,B,C}(\psi)\right] = \sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\sum_{j=1}^{n_3} \psi(\lambda_i^{(1)}, \lambda_k^{(2)}, \lambda_j^{(3)}) E_i^{(1)} X E_k^{(2)} Y E_j^{(3)} \tag{4.18}$$

for any $X, Y \in \mathcal{B}(\mathbb{C}^n)$. The proof of this formula is similar to the one of Formula (4.15).

Let us consider two different identifications of $\mathcal{B}(\mathbb{C}^n)$ with M_n. On one hand, we identify X with the matrix $\{x_{ik}^{\xi_1,\xi_2}\}_{i,k=1}^n$, where $x_{ik}^{\xi_1,\xi_2} = \langle X(\xi_k^{(2)}), \xi_i^{(1)} \rangle$. On the other hand we identify Y with $\{y_{kj}^{\xi_2,\xi_3}\}_{k,j=1}^n$, where $y_{kj}^{\xi_2,\xi_3} = \langle Y(\xi_j^{(3)}), \xi_k^{(2)}\rangle$. Under these identifications, the operator $\Gamma^{A,B,C}(\psi)$ acts as a bilinear Schur multiplier associated with the matrix $M = \{\psi(\lambda_i^{(1)}, \lambda_k^{(2)}, \lambda_j^{(3)})\}_{i,j,k=1}^n$. Indeed,

$$\big\langle (P_{\xi_i^{(1)}} X P_{\xi_k^{(2)}} Y P_{\xi_j^{(3)}})(\xi_{\mathcal{S}}^{(3)}), \xi_r^{(1)}\big\rangle = \big\langle Y(\xi_s^{(3)}), \xi_k^{(2)}\big\rangle \big\langle X(\xi_k^{(2)}), \xi_r^{(1)}\big\rangle = y_{ks}^{\xi_2,\xi_3} x_{rk}^{\xi_1,\xi_2}$$

if $s = j, r = i$, and

$$\big\langle (P_{\xi_i^{(1)}} X P_{\xi_k^{(2)}} Y P_{\xi_j^{(3)}})(\xi_s^{(3)}), \xi_r^{(1)}\big\rangle = 0$$

otherwise.

Therefore,

$$\big\langle \left[\Gamma^{A,B,C}(\psi)\right](X,Y)(\xi_s^{(3)}), \xi_r^{(1)}\big\rangle = \sum_{k=1}^n \psi(\lambda_r^{(1)}, \lambda_k^{(2)}, \lambda_s^{(3)}) y_{ks}^{\xi_2,\xi_3} x_{rk}^{\xi_1,\xi_2},$$

which implies

$$\left[\Gamma^{A,B,C}(\psi)\right](X,Y) = \sum_{i,j,k=1}^n \psi(\lambda_i^{(1)}, \lambda_k^{(2)}, \lambda_j^{(3)}) x_{ik}^{\xi_1,\xi_2} y_{kj}^{\xi_2,\xi_3} E_{ij}^{\xi_1,\xi_3}.$$

Since these identifications are isometric ones with respect to all Schatten norms, we deduce the formula

$$\big\|\Gamma^{A,B,C}(\psi) \colon \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\big\| = \big\|\{\psi(\lambda_i^{(1)}, \lambda_k^{(2)}, \lambda_j^{(3)})\}_{i,j,k=1}^n \colon \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\big\|. \tag{4.19}$$

4.3 Characterization of $\mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1$ boundedness

Let $\mathcal{H}$ be a separable Hilbert space and let A, B and C be normal operators on $\mathcal{H}$. Let λ_A, λ_B and λ_C be scalar-valued spectral measures associated with A, B and C. Recall the definition of the triple operator mapping $\Gamma^{A,B,C}$ from Theorem 4.3. The purpose of this section is to characterize the functions $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$ such that $\Gamma^{A,B,C}(\phi)$ maps $\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H})$ into $\mathcal{S}^1(\mathcal{H})$.

We shall start with a factorization formula of independent interest. Let $\Gamma^{A,B}$ and $\Gamma^{B,C}$ be the double operator integral mappings associated respectively with (A, B) and with (B, C), see Proposition 4.4. It is important to note that $\Gamma^{A,B}$ and $\Gamma^{B,C}$ are $*$-representations. Recall that they are w^*-continuous.

Lemma 4.9. *Let $u \in L^\infty(\lambda_A \times \lambda_B)$ and $v \in L^\infty(\lambda_B \times \lambda_C)$. Then, for all $X, Y \in \mathcal{S}^2(\mathcal{H})$, we have*

$$\Gamma^{A,B,C}(uv)(X,Y) = \Gamma^{A,B}(u)(X)\Gamma^{B,C}(v)(Y).$$

Proof. Fix $X, Y \in \mathcal{S}^2(\mathcal{H})$. Let $u_1 \in L^\infty(\lambda_A), u_2, v_1 \in L^\infty(\lambda_B)$ and $v_2 \in L^\infty(\lambda_C)$. Consider $u = u_1 \otimes u_2 \in L^\infty(\lambda_A) \otimes L^\infty(\lambda_B)$ and $v = v_1 \otimes v_2 \in L^\infty(\lambda_B) \otimes L^\infty(\lambda_C)$. Then we have

$uv = u_1 \otimes u_2 v_1 \otimes v_2 \in L^\infty(\lambda_A) \otimes L^\infty(\lambda_B) \otimes L^\infty(\lambda_C)$. Therefore

$$\begin{aligned}\Gamma^{A,B,C}(uv)(X,Y) &= u_1(A)X(u_2v_1)(B)Yv_2(C)\\ &= u_1(A)Xu_2(B)v_1(B)Yv_2(C)\\ &= \Gamma^{A,B}(u)(X)\Gamma^{B,C}(v)(Y).\end{aligned}$$

Now, take $u \in L^\infty(\lambda_A \times \lambda_B)$ and $v \in L^\infty(\lambda_B \times \lambda_C)$. Let $(u_i)_i$ and $(v_j)_j$ be two nets in $L^\infty(\lambda_A) \otimes L^\infty(\lambda_B)$ and $L^\infty(\lambda_B) \otimes L^\infty(\lambda_C)$ respectively, converging to u and v in the w^*-topology. By linearity, the previous calculation implies that for all i, j,

$$\Gamma^{A,B,C}(u_iv_j)(X,Y) = \Gamma^{A,B}(u_i)(X)\Gamma^{B,C}(v_j)(Y).$$

Take $Z \in \mathcal{S}^2(\mathcal{H})$ and fix j. Since $\Gamma^{B,C}(v_j)(Y)Z$ belongs to $\mathcal{S}^2(\mathcal{H})$ we have

$$\begin{aligned}\lim_i \mathrm{tr}(\Gamma^{A,B}(u_i)(X)\Gamma^{B,C}(v_j)(Y)Z) &= \mathrm{tr}(\Gamma^{A,B}(u)(X)\Gamma^{B,C}(v_j)(Y)Z)\\ &= \mathrm{tr}(\Gamma^{B,C}(v_j)(Y)Z\Gamma^{A,B}(u)(X))\end{aligned}$$

by the w^*-continuity of $\Gamma^{A,B}$. Similarly, since $Z\Gamma^{A,B}(u)(X) \in \mathcal{S}^2(\mathcal{H})$, the w^*-continuity of $\Gamma^{B,C}$ implies that

$$\begin{aligned}\lim_j \mathrm{tr}(\Gamma^{B,C}(v_j)(Y)Z\Gamma^{A,B}(u)(X)) &= \mathrm{tr}(\Gamma^{B,C}(v)(Y)Z\Gamma^{A,B}(u)(X))\\ &= \mathrm{tr}(\Gamma^{A,B}(u)(X)\Gamma^{B,C}(v)(Y)Z).\end{aligned}$$

On the other hand, $(u_iv_j)_i$ w^*-converges to uv_j for any fixed j and $(uv_j)_j$ w^*-converges to uv in $L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$. Hence the w^*-continuity of $\Gamma^{A,B,C}$ implies that

$$\begin{aligned}\lim_j \lim_i \mathrm{tr}(\Gamma^{A,B,C}(u_iv_j)(X,Y)Z) &= \lim_j \mathrm{tr}(\Gamma^{A,B,C}(uv_j)(X,Y)Z)\\ &= \mathrm{tr}(\Gamma^{A,B,C}(uv)(X,Y)Z).\end{aligned}$$

Thus, for all $Z \in \mathcal{S}^2(\mathcal{H})$,

$$\mathrm{tr}(\Gamma^{A,B}(u)(X)\Gamma^{B,C}(v)(Y)Z) = \mathrm{tr}(\Gamma^{A,B,C}(uv)(X,Y)Z),$$

which implies that $\Gamma^{A,B,C}(uv) = \Gamma^{A,B}(u)(X)\Gamma^{B,C}(v)(Y)$. □

Our main result is the following theorem. In this statement, as in Subsection 4.1.3, we consider the continuous bilinear Schur multipliers $\Lambda(\phi)$ in the case when $(\Omega_1, \mu_1) = (\sigma(C), \lambda_C)$, $(\Omega_2, \mu_2) = (\sigma(B), \lambda_B)$ and $(\Omega_3, \mu_3) = (\sigma(A), \lambda_A)$. Note that these measurable spaces are separable.

Theorem 4.10. *Let $\mathcal{H}$ be a separable Hilbert space, let A, B and C be normal operators on $\mathcal{H}$ and let $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$. The following are equivalent :*

(i) $\Gamma^{A,B,C}(\phi) \in \mathcal{B}_2(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}), \mathcal{S}^1(\mathcal{H}))$.

(ii) $\Lambda(\phi) \in \mathcal{B}_2(\mathcal{S}^2(L^2(\lambda_C), L^2(\lambda_B)) \times \mathcal{S}^2(L^2(\lambda_B), L^2(\lambda_A)), \mathcal{S}^1(L^2(\lambda_C), L^2(\lambda_A)))$.

(iii) There exist a Hilbert space H and two functions

$$a \in L^\infty(\lambda_A \times \lambda_B; H) \qquad \text{and} \qquad b \in L^\infty(\lambda_B \times \lambda_C; H)$$

such that $\|a\|_\infty \|b\|_\infty \le \|\phi\|_{\infty,\Gamma_2}$ *and*

$$\phi(t_1, t_2, t_3) = \langle a(t_1, t_2), b(t_2, t_3)\rangle$$

for a.-e. $(t_1, t_2, t_3) \in \sigma(A) \times \sigma(B) \times \sigma(C)$.

In this case,

$$\left\|\Gamma^{A,B,C}(\phi)\colon \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\right\| = \left\|\Lambda(\phi)\colon \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\right\| = \inf \|a\|_\infty \|b\|_\infty. \tag{4.20}$$

Proof. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 3.8.

Proof of (iii) $\Rightarrow$ (i)

Assume (iii) and let $(\epsilon_k)_{k\in\mathbb{N}}$ be a Hilbertian basis of H. For any $k \in \mathbb{N}$, define

$$a_k = \langle a, \epsilon_k\rangle \in L^\infty(\lambda_A \times \lambda_B) \qquad \text{and} \qquad b_k = \langle b, \epsilon_k\rangle \in L^\infty(\lambda_B \times \lambda_C).$$

We set

$$|a| = \Big(\sum_n |a_k|^2\Big)^{\frac{1}{2}};$$

this function belongs to $L^\infty(\lambda_A \times \lambda_B)$ and we have $\|a\|_\infty = \||a|\|_\infty$.

Let $X \in \mathcal{S}^2(\mathcal{H})$. Since $\Gamma^{A,B}$ is a w^*-continuous $*$-representation, we have

$$\begin{aligned}
\sum_k \|\Gamma^{A,B}(a_k)(X)\|_2^2 &= \sum_k \big\langle \Gamma^{A,B}(a_k)(X), \Gamma^{A,B}(a_k)(X)\big\rangle \\
&= \sum_n \big\langle \Gamma^{A,B}(\overline{a_k})\Gamma^{A,B}(a_k)(X), X\big\rangle \\
&= \big\langle \Gamma^{A,B}(|a|^2)(X), X\big\rangle \\
&\le \||a|^2\|_\infty \|X\|_2^2 = \|a\|_\infty^2 \|X\|_2^2.
\end{aligned}$$

We prove similarly that if $Y \in \mathcal{S}^2(\mathcal{H})$, then

$$\sum_n \|\Gamma^{B,C}(\overline{b_k})(Y)\|_2^2 \le \|b\|_\infty^2 \|Y\|_2^2.$$

Consequently, for all $X, Y \in \mathcal{S}^2(\mathcal{H})$, we have the inequalities

$$\begin{aligned}
\sum_k \|\Gamma^{A,B}(a_k)(X)\Gamma^{B,C}(\overline{b_k})(Y)\|_1 &\le \sum_k \|\Gamma^{A,B}(a_k)(X)\|_2 \|\Gamma^{B,C}(\overline{b_k})(Y)\|_2 \\
&\le \Big(\sum_k \|\Gamma^{A,B}(a_k)(X)\|_2^2\Big)^{1/2} \Big(\sum_k \|\Gamma^{B,C}(\overline{b_k})(Y)\|_2^2\Big)^{1/2} \\
&\le \|a\|_\infty \|b\|_\infty \|X\|_2 \|Y\|_2.
\end{aligned}$$

Therefore, we can define a bounded bilinear map

$$\Theta\colon \mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}) \longrightarrow \mathcal{S}^1(\mathcal{H})$$

by

$$\Theta(X,Y) = \sum_{k=1}^{\infty} \Gamma^{A,B}(a_k)(X)\Gamma^{B,C}(\overline{b_k})(Y), \qquad X, Y \in \mathcal{S}^2(\mathcal{H}),$$

and we have

$$\|\Theta\| \leq \|a\|_\infty \|b\|_\infty. \tag{4.21}$$

We claim that

$$\Gamma^{A,B,C}(\phi) = \Theta.$$

To check this, consider

$$\widetilde{a_n} = \sum_{k=0}^{n} a_k \otimes \epsilon_k \qquad \text{and} \qquad \widetilde{b_n} = \sum_{k=0}^{n} b_k \otimes \epsilon_k$$

for any $n \in \mathbb{N}$. Then we set

$$\phi_n(t_1,t_2,t_3) = \left\langle \widetilde{a_n}(t_1,t_2), \widetilde{b_n}(t_2,t_3) \right\rangle = \sum_{k=0}^{n} a_k(t_1,t_2)\overline{b_k(t_2,t_3)}.$$

Fix $X, Y \in \mathcal{S}^2(\mathcal{H})$. We have $\Gamma^{A,B,C}(\phi_n) = \sum_{k=0}^{n} \Gamma^{A,B,C}(a_k\overline{b_k})$ hence by Lemma 4.9,

$$\Gamma^{A,B,C}(\phi_n)(X,Y) = \sum_{k=0}^{n} \Gamma^{A,B}(a_k)(X)\Gamma^{B,C}(\overline{b_k})(Y).$$

Consequently,

$$\Gamma^{A,B,C}(\phi_n)(X,Y) \underset{n\to+\infty}{\longrightarrow} \Theta(X,Y) \text{ in } \mathcal{S}^1(\mathcal{H}).$$

Moreover $\phi_n \to \phi$ a.-e. and $(\phi_n)_n$ is bounded in $L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$. Indeed,

$$\left|\phi_n(t_1,t_2,t_3)\right| \leq \left(\sum_{k=0}^{n} |a_k(t_1,t_2)|^2\right)^{\frac{1}{2}} \left(\sum_{k=0}^{n} |b_k(t_2,t_3)|^2\right)^{\frac{1}{2}} \leq \|a\|_\infty \|b\|_\infty.$$

Hence by Lebesgue's dominated convergence Theorem, $w^*\text{-}\lim\limits_{n\to+\infty} \phi_n = \phi$. The w^*-continuity of $\Gamma^{A,B,C}$ implies that

$$\Gamma^{A,B,C}(\phi_n)(X,Y) \underset{n\to+\infty}{\longrightarrow} \Gamma^{A,B,C}(\phi)(X,Y) \text{ weakly in } \mathcal{S}^2(\mathcal{H}).$$

We conclude that $\Gamma^{A,B,C}(\phi)(X,Y) = \Theta(X,Y)$.

This shows (i). Furthermore (4.21) yields

$$\left\|\Gamma^{A,B,C}(\phi)\colon \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\right\| \leq \|a\|_\infty \|b\|_\infty \tag{4.22}$$

Proof of (i) $\Rightarrow$ (ii)

Assume (i) and apply Proposition 4.7, which connects $\Gamma^{A,B,C}(\phi)$ to $\Lambda(\phi)$. Let $X \in \mathcal{S}^2(L^2(\lambda_B), L^2(\lambda_A))$ and $Y \in \mathcal{S}^2(L^2(\lambda_C), L^2(\lambda_B))$. By (4.10), we have

$$\|\Lambda(\phi)(Y,X)\|_1 = \|\rho_A^{-1} \circ \Gamma^{A,B,C}(\phi)(\widetilde{X},\widetilde{Y}) \circ \rho_C\|_1$$

$$\leq \|\Gamma^{A,B,C}(\phi)(\widetilde{X},\widetilde{Y})\|_1$$
$$\leq \big\|\Gamma^{A,B,C}(\phi)\colon \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\big\| \|X\|_2 \|Y\|_2,$$

since $\|\widetilde{X}\|_2 = \|X\|_2$ and $\|\widetilde{Y}\|_2 = \|Y\|_2$. This shows (ii), with

$$\big\|\Lambda(\phi)\colon \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\big\| \leq \big\|\Gamma^{A,B,C}(\phi)\colon \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\big\|. \tag{4.23}$$

□

Remark 4.11. *With the terminology adopted here, Peller's Theorem from [Pel85] states as follows.*

Let A, B be normal operators on a separable Hilbert space $\mathcal{H}$ and let λ_A and λ_B be scalar-valued spectral measures for A and B. Let $\psi \in L^\infty(\lambda_A \times \lambda_B)$ and let $u_\psi \colon L^1(\lambda_A) \to L^\infty(\lambda_B)$ be the bounded map associated to ψ (see (1.5)). The following are equivalent.

(i) The double operator integral mapping $\Gamma^{A,B}(\psi)$ extends to a bounded map from $\mathcal{S}^1(\mathcal{H})$ into itself.

(ii) There exist a Hilbert space H and two functions $a \in L^\infty(\lambda_A; H)$ and $b \in L^\infty(\lambda_B; H)$ such that
$$\varphi(s,t) = \langle a(s), b(t)\rangle \qquad a.e.\text{-}(s,t). \tag{4.24}$$

In this case,
$$\big\|\Gamma^{A,B}(\varphi)\colon \mathcal{S}^1(\mathcal{H}) \longrightarrow \mathcal{S}^1(\mathcal{H})\big\| = \inf\big\{\|a\|_\infty \|b\|_\infty\big\},$$
where the infimum runs over all pairs (a,b) of functions such that (4.24) holds true.

Let us show that this result directly follows from Theorem 4.10. Consider A, B as above and take an auxiliary normal operator C on $\mathcal{H}$ (this may be the identity map), with a scalar-valued spectral measure λ_C. For any $\psi \in L^\infty(\lambda_A \times \lambda_B)$, set

$$\widetilde{\psi} = \psi \otimes 1 \in L^\infty(\lambda_A \times \lambda_B) \otimes L^\infty(\lambda_C) \subset L^\infty(\lambda_A \times \lambda_C \times \lambda_B).$$

We claim that for any $X, Y \in \mathcal{S}^2(\mathcal{H})$,

$$\Gamma^{A,C,B}(\widetilde{\psi})(X,Y) = \Gamma^{A,B}(\psi)(XY). \tag{4.25}$$

Indeed for any $f_1 \in L^\infty(\lambda_A)$ and $f_2 \in L^\infty(\lambda_B)$, and for any $X, Y \in \mathcal{S}^2(\mathcal{H})$, we have

$$\Gamma^{A,C,B}(f_1 \otimes 1 \otimes f_2)(X,Y) = f_1(A)XYf_2(B).$$

Hence by linearity, (4.25) holds true for any $\psi \in L^\infty(\lambda_A) \otimes L^\infty(\lambda_B)$. By the w^*-continuity of $\Gamma^{A,C,B}$ and of $\Gamma^{A,B}$, this identity holds as well for any $\psi \in L^\infty(\lambda_A \times \lambda_B)$.

We have $\|XY\|_1 \leq \|X\|_2\|Y\|_2$ for any $X, Y \in \mathcal{S}^2(\mathcal{H})$ and conversely, for any $Z \in \mathcal{S}^1(\mathcal{H})$, there exist X, Y in $\mathcal{S}^2(\mathcal{H})$ such that $XY = Z$ and $\|X\|_2\|Y\|_2 = \|Z\|_1$. Thus given any $\psi \in L^\infty(\lambda_A \times \lambda_B)$, it follows from (4.25) that $\Gamma^{A,C,B}(\widetilde{\psi})$ maps $\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H})$ into $\mathcal{S}^1(\mathcal{H})$ if and only if $\Gamma^{A,B}(\psi)$ maps $\mathcal{S}^1(\mathcal{H})$ into $\mathcal{S}^1(\mathcal{H})$ and moreover,

$$\big\|\Gamma^{A,C,B}(\widetilde{\psi})\colon \mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}) \longrightarrow \mathcal{S}^1(\mathcal{H})\big\| = \big\|\Gamma^{A,B}(\psi)\colon \mathcal{S}^1(\mathcal{H}) \longrightarrow \mathcal{S}^1(\mathcal{H})\big\|.$$

The result therefore follows from Theorem 4.10 and the equality (4.20).

4.4 Complete boundedness of triple operator integrals

Let A, B, C be normal operators on a separable Hilbert space $\mathcal{H}$. Let λ_A, λ_B and λ_C be scalar-valued spectral measures associated with A, B and C. The purpose of this section is to characterize the functions $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$ such that $\Gamma^{A,B,C}(\phi)$ extends to a completely bounded map from $\mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H})$ into $\mathcal{S}^\infty(\mathcal{H})$.

We will also consider the continuous bilinear Schur multipliers $\Lambda(\phi)$. Note that by the obvious equalities

$$\mathcal{S}^2(L^2(\lambda_B), L^2(\lambda_C)) = \mathcal{S}^2(L^2(\lambda_C), L^2(\lambda_B))$$

and

$$\mathcal{S}^2(L^2(\lambda_A), L^2(\lambda_B)) = \mathcal{S}^2(L^2(\lambda_B), L^2(\lambda_A)),$$

we can see $\Lambda(\phi)$ as a mapping

$$\Lambda(\phi) : \mathcal{S}^2(L^2(\lambda_B), L^2(\lambda_C)) \times \mathcal{S}^2(L^2(\lambda_A), L^2(\lambda_B)) \to \mathcal{S}^2(L^2(\lambda_A), L^2(\lambda_C)).$$

In [KJT09], the authors studied and characterized the boundedness of continuous bilinear Schur multipliers

$$\mathcal{S}^\infty(L^2(\lambda_B), L^2(\lambda_C)) \overset{h}{\otimes} \mathcal{S}^\infty(L^2(\lambda_A), L^2(\lambda_B)) \to \mathcal{S}^\infty(L^2(\lambda_A), L^2(\lambda_C)).$$

They proved that we have such extension if and only if ϕ has a certain factorization that will be given in the theorem of this section. They also proved that the boundedness for the Haagerup norm in this setting implies the complete boundedness.

The proof of Theorem 4.12 below includes another proof of [KJT09, Theorem 3.4] and we show that the same characterization holds for triple operator integrals. Note that the result presented here can be easily extended to the case of multilinear operator integrals.

Theorem 4.12. *Let $\mathcal{H}$ be a separable Hilbert space, A, B, C be normal operators on $\mathcal{H}$ and let $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$. The following are equivalent:*

(i) $\Gamma^{A,B,C}(\phi)$ *extends to a completely bounded mapping*

$$\Gamma^{A,B,C}(\phi) : \mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}) \to \mathcal{S}^\infty(\mathcal{H}).$$

(ii) $\Lambda(\phi)$ *extends to a completely bounded mapping*

$$\Lambda(\phi) : \mathcal{S}^\infty(L^2(\lambda_B), L^2(\lambda_C)) \overset{h}{\otimes} \mathcal{S}^\infty(L^2(\lambda_A), L^2(\lambda_B)) \to \mathcal{S}^\infty(L^2(\lambda_A), L^2(\lambda_C)).$$

(iii) There exist a separable Hilbert space H, $a \in L^\infty(\lambda_A; H), b \in L^\infty_\sigma(\lambda_B; \mathcal{B}(H))$ and $c \in L^\infty(\lambda_C; H)$ such that

$$\phi(t_1, t_2, t_3) = \langle [b(t_2)](a(t_1)), c(t_3) \rangle$$

for a.-e. $(t_1, t_2, t_3) \in \sigma(A) \times \sigma(B) \times \sigma(C)$.

In this case,

$$\left\| \Gamma^{A,B,C}(\phi) \right\| = \|\Lambda(\phi)\| = \inf \|a\|_\infty \|b\|_\infty \|c\|_\infty. \tag{4.26}$$

Proof. In this proof we will identify, for $\psi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$, the element $\Gamma^{A,B,C}(\psi) \in \mathcal{B}_2(\mathcal{S}^2(\mathcal{H}) \times \mathcal{S}^2(\mathcal{H}), \mathcal{S}^2(\mathcal{H}))$ with the element still denoted by

$$\Gamma^{A,B,C}(\psi) : \mathcal{S}^2(\mathcal{H}) \overset{\wedge}{\otimes} \mathcal{S}^2(\mathcal{H}) \to \mathcal{S}^2(\mathcal{H}).$$

(See the isometry given by (1.1).)

Proof of (i) $\Rightarrow$ (ii)

We use the same notations as in Subsection 4.1.3 where we introduced the subspaces $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_C$ of $\mathcal{H}$. $\mathcal{S}^\infty(\mathcal{H}_B, \mathcal{H}_C)$ and $\mathcal{S}^\infty(\mathcal{H}_A, \mathcal{H}_B)$ are closed subspaces of $\mathcal{S}^\infty(\mathcal{H})$ and by injectivity of the Haagerup tensor product (see Proposition 1.4), we have a closed subspace

$$\mathcal{S}^\infty(\mathcal{H}_B, \mathcal{H}_C) \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}_A, \mathcal{H}_B) \subset \mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}).$$

By Proposition 4.7, the restriction of $\Gamma^{A,B,C}(\phi)$ to $\mathcal{S}^\infty(\mathcal{H}_B, \mathcal{H}_C) \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}_A, \mathcal{H}_B)$ is valued in $\mathcal{S}^\infty(\mathcal{H}_A, \mathcal{H}_C)$. Moreover, this restriction is completely bounded and by the same proposition, we obtain the inequality

$$\|\Lambda(\phi)\|_{\text{cb}} \leq \left\|\Gamma^{A,B,C}(\phi)\right\|_{\text{cb}}.$$

Proof of (ii) $\Rightarrow$ (iii)

If (Ω, μ) is a measure space, the mapping

$$(f, g) \in L^2(\Omega)^2 \mapsto fg \in L^1(\Omega)$$

induces a quotient map

$$f \otimes g \in L^2(\Omega) \overset{\wedge}{\otimes} L^2(\Omega) \mapsto fg \in L^1(\Omega).$$

We can identify $L^2(\Omega)$ with its conjugate space so that by (1.12) we get a quotient map

$$q : \mathcal{S}^1(L^2(\Omega)) \to L^1(\Omega)$$

which turns out to be a complete metric surjection (here, the L^1-spaces are equipped with their Max structure).

Let $q_i : \mathcal{S}^1(L^2(\Omega_i)) \to L^1(\Omega_i), i = 1, 2, 3$ be defined as above. For convenience, write $H_i = L^2(\Omega_i)$. Using Proposition 1.4 together with the associativity of the Haagerup tensor product, we get a complete metric surjection

$$Q = q_3 \otimes q_2 \otimes q_1 : \mathcal{S}^1(H_3) \overset{h}{\otimes} \mathcal{S}^1(H_2) \overset{h}{\otimes} \mathcal{S}^1(H_1) \to L^1(\Omega_3) \overset{h}{\otimes} L^1(\Omega_2) \overset{h}{\otimes} L^1(\Omega_1).$$

Let $N = \ker Q$ and let, for $i = 1, 2, 3$, $N_i = \ker q_i$. Using Corollary 1.5 twice, we obtain that

$$N = \overline{N_3 \otimes \mathcal{S}^1(H_2) \otimes \mathcal{S}^1(H_1) + \mathcal{S}^1(H_3) \otimes N_2 \otimes \mathcal{S}^1(H_1) + \mathcal{S}^1(H_3) \otimes \mathcal{S}^1(H_2) \otimes N_1}.$$

Assume that $\Lambda(\phi)$ extends to a completely bounded mapping

$$\Lambda(\phi) : \mathcal{S}^\infty(H_2, H_3) \overset{h}{\otimes} \mathcal{S}^\infty(H_1, H_2) \to \mathcal{S}^\infty(H_1, H_3) \subset \mathcal{B}(H_1, H_3).$$

Let $E = \mathcal{S}^\infty(H_2, H_3) \overset{h}{\otimes} \mathcal{S}^\infty(H_1, H_2)$. By Proposition 1.6, we have a complete isometry

$$CB(E, \mathcal{B}(H_1, H_3)) = \left(((\mathcal{H}_3)_c)^* \overset{h}{\otimes} E \overset{h}{\otimes} (\mathcal{H}_1)_c \right)^* .$$

By (1.14) we have

$$E = (\mathcal{H}_3)_c \overset{h}{\otimes} ((\mathcal{H}_2)_c)^* \overset{h}{\otimes} (\mathcal{H}_2)_c \overset{h}{\otimes} ((\mathcal{H}_1)_c)^*.$$

Thus, using (1.13) and the associativity of the Haagerup tensor product, we get that

$$CB(E, \mathcal{B}(H_1, H_3)) = \left(\mathcal{S}^1(H_3) \overset{h}{\otimes} \mathcal{S}^1(H_2) \overset{h}{\otimes} \mathcal{S}^1(H_1) \right)^* .$$

Let $u : \mathcal{S}^1(H_3) \overset{h}{\otimes} \mathcal{S}^1(H_2) \overset{h}{\otimes} \mathcal{S}^1(H_1) \to \mathbb{C}$ induced by $\Lambda(\phi)$. For any $x \in \mathcal{S}^1(H_1), y \in \mathcal{S}^1(H_2)$ and $z \in \mathcal{S}^1(H_3)$, we have

$$u(z \otimes y \otimes x) = \int_{\Omega_1 \times \Omega_2 \times \Omega_3} \phi(t_1, t_2, t_3)[q_1(x)](t_1)[q_2(y)](t_2)[q_3(z)](t_3)\, \mathrm{d}\mu_1(t_1)\mathrm{d}\mu_2(t_2)\mathrm{d}\mu_3(t_3).$$

To see this, it is enough to check it when x, y and z are rank one operators and in that case, one can use the identifications above. In particular, the latter implies that u vanishes on $N = \ker Q$. Since Q is a complete metric surjection, we get a mapping

$$v : L^1(\Omega_3) \overset{h}{\otimes} L^1(\Omega_2) \overset{h}{\otimes} L^1(\Omega_1) \to \mathbb{C}$$

such that $u = v \circ Q$. An application of Theorem 1.1 with suitable restrictions using the separability of the spaces $L^1(\Omega_i)$ gives the existence of a separable Hilbert space H and completely bounded maps

$$\alpha : L^1(\Omega_1) \to H_c, \beta : L^1(\Omega_2) \to \mathcal{B}(H) \text{ and } \gamma : L^1(\Omega_3) \to H_r$$

such that for any $f \in L^1(\Omega_1), g \in L^1(\Omega_2), h \in L^1(\Omega_3)$,

$$v(h \otimes g \otimes f) = \langle [\beta(g)](\alpha(f)), \gamma(h) \rangle .$$

Since $L^1(\Omega_2)$ is equipped with the Max operator space structure, we have

$$CB(L^1(\Omega_2), \mathcal{B}(H)) = \mathcal{B}(L^1(\Omega_2), \mathcal{B}(H)).$$

Moreover, by (1.4) and (1.16), we have

$$\mathcal{B}(L^1(\Omega_2), \mathcal{B}(H)) = L^\infty_\sigma(\Omega_2; \mathcal{B}(H)).$$

Thus, we associate to β an element $b \in L^\infty_\sigma(\Omega_2; \mathcal{B}(H))$. Similarly, we associate to α an element $a \in L^\infty(\Omega_1; H)$ and to γ an element $c \in L^\infty(\Omega_3; H)$. We obtain that

$$\phi(t_1, t_2, t_3) = \langle [b(t_2)](a(t_1)), c(t_3) \rangle$$

for a.-e. $(t_1, t_2, t_3) \in \sigma(A) \times \sigma(B) \times \sigma(C)$, and one can choose a, b and c such that we have the equality

$$\|\Lambda(\phi)\| = \|a\|_\infty \|b\|_\infty \|c\|_\infty.$$

Proof of (iii) $\Rightarrow$ (i)

Assume that there exist a separable Hilbert space H, $a \in L^\infty(\lambda_A; H)$, $b \in L^\infty_\sigma(\lambda_B; \mathcal{B}(H))$ and $c \in L^\infty(\lambda_C; H)$ such that

$$\phi(t_1, t_2, t_3) = \langle [b(t_2)](a(t_1)), c(t_3) \rangle$$

for a.-e. $(t_1, t_2, t_3) \in \sigma(A) \times \sigma(B) \times \sigma(C)$. Let $(\epsilon_n)_{n \geq 1}$ be a Hilbertian basis of H. Define, for $i, j \geq 1$,

$$a_i = \langle a, \epsilon_i \rangle, b_{ij} = \langle b\epsilon_j, \epsilon_i \rangle \quad \text{and} \quad c_j = \langle \epsilon_j, c \rangle.$$

Then $a \in L^\infty(\Omega_1)$, $c \in L^\infty(\Omega_3)$ and $b \in L^\infty(\Omega_2)$. To see this last point, simply note that

$$b_{ij} = \overline{\text{tr}(b(.) \circ (\epsilon_i \otimes \epsilon_j))}.$$

For $N \geq 1$, let P_N be the orthogonal projection onto Span$(\epsilon_1, \ldots, \epsilon_N)$. Then, define

$$\phi_N(t_1, t_2, t_3) = \langle [b(t_2)](P_N(a(t_1))), P_N(c(t_3)) \rangle.$$

It is clear that $(\phi_N)_{N \geq 1}$ is bounded in $L^\infty(\lambda_1 \times \lambda_B \times \lambda_C)$ and that $\phi_N \to \phi$ pointwise when $N \to \infty$. Therefore, by Dominated convergence theorem, we have that $\phi_N \to \phi$ for the w^*-topology. This implies, by w^*- continuity of $\Gamma^{A,B,C}$, that for any X and Y in $\mathcal{S}^2(\mathcal{H})$,

$$\left[\Gamma^{A,B,C}(\phi_N)\right](X \otimes Y) \to \left[\Gamma^{A,B,C}(\phi)\right](X \otimes Y)$$

weakly in $\mathcal{S}^2(\mathcal{H})$.

Assume that $(\Gamma^{A,B,C}(\phi_N))_N$ is uniformly bounded in $CB(\mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}), \mathcal{S}^\infty(\mathcal{H}))$. Then, the above approximation property together with the density of $\mathcal{S}^2$ into $\mathcal{S}^\infty$ imply that $\Gamma^{A,B,C}(\phi)$ is completely bounded as well.

We will show now that for any $N \geq 1$, $\Gamma^{A,B,C}(\phi_N) \in CB(\mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}), \mathcal{S}^\infty(\mathcal{H}))$ with a cb-norm less than $\|a\|_\infty \|b\|_\infty \|c\|_\infty$.

For any $N \geq 1$ and a.-e. $(t_1, t_2, t_3) \in \sigma(A) \times \sigma(B) \times \sigma(C)$, we have

$$\phi_N(t_1, t_2, t_3) = \sum_{1 \leq n \leq N} \left(\sum_{1 \leq m \leq N} a_m(t_1) b_{nm}(t_2) \right) c_n(t_3),$$

so that for any $X, Y \in \mathcal{S}^2(\mathcal{H})$,

$$\left[\Gamma^{A,B,C}(\phi_N)\right](X \otimes Y) = \sum_{1 \leq n \leq N} \left(\sum_{1 \leq m \leq N} a_m(A) X b_{nm}(B) \right) Y c_n(C).$$

Note that the latter can be written as

$$\left[\Gamma^{A,B,C}(\phi_N)\right](X\otimes Y) = A_N(X\otimes I_N)B_N(Y\otimes I_N)C_N,$$

where

$$A_N = (a_1(A)\, a_2(A) \ldots a_N(A)) : \ell_2^N(\mathcal{H}) \to \mathcal{H},$$

$$B_N = (b_{ji}(B))_{\substack{1\leq i\leq N\\ 1\leq j\leq N}} : \ell_2^N(\mathcal{H}) \to \ell_2^N(\mathcal{H})$$

and

$$C_N = (c_1(C)\, c_2(C) \ldots c_N(C))^t : \mathcal{H} \to \ell_2^N(\mathcal{H}).$$

The notation $X\otimes I_N$ stands for the element of $\mathcal{B}(\ell_2^N(\mathcal{H}))$ whose matrix is the $N\times N$ diagonal matrix diag$(X,\ldots,X)$. Similarly for $Y\otimes I_N$.

Define, for $N\geq 1$,

$$\begin{array}{rccc}\pi_N : & \mathcal{B}(\mathcal{H}) & \longrightarrow & \mathcal{B}(\ell_2^N(\mathcal{H})). \\ & X & \longmapsto & X\otimes I_N\end{array}$$

Then π_N is a $*-$representation.

Let

$$\begin{array}{rccc}\pi_B : & L^\infty(\lambda_B) & \longrightarrow & \mathcal{B}(\mathcal{H}) \\ & f & \longmapsto & f(B)\end{array}$$

be the $*$-representation introduced in Subsection 4.1.1. By [Pis03, Proposition 1.5], π_B is completely bounded and $\|\pi_B\|_{\text{cb}}\leq 1$. Note that the element

$$(b_{ji})_{\substack{1\leq i\leq N\\ 1\leq j\leq N}} \in M_N(L^\infty(\lambda_B))$$

has a norm less than $\|b\|_\infty$. Thus, the latter implies that

$$B_N = (\pi_B(b_{ji}))_{\substack{1\leq i\leq N\\ 1\leq j\leq N}}$$

has an operator norm less than $\|b\|_\infty$. Similarly (using column and row matrices), we show that A_N and C_N have a norm less than $\|a\|_\infty$ and $\|c\|_\infty$, respectively. Finally, write

$$\left[\Gamma^{A,B,C}(\phi_N)\right](X\otimes Y) = \sigma_1^N(X)\sigma_2^N(Y)$$

where $\sigma_1^N(X) = A_N\pi(X)B_N$ and $\sigma_2^N(Y) = \pi(Y)C_N$. By the easy part of Wittstock theorem (see e.g. [Pis03, Theorem 1.6]), σ_1^N and σ_2^N are completely bounded with cb-norm less than $\|a\|_\infty\|b\|_\infty$ and $\|c\|_\infty$, respectively. By Theorem 1.1, we obtain that $\Gamma^{A,B,C}(\phi_N)$ belongs to $CB(\mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}), \mathcal{S}^\infty(\mathcal{H}))$ with cb-norm less than $\inf \|a\|_\infty\|b\|_\infty\|c\|_\infty$. This completes the proof of the theorem.

□

Remark 4.13. *In the theorem above, note that the implication $(i)\Rightarrow(ii)$ holds true when we replace 'complete boundedness' by 'boundedness'. In [KJT09], it is proved that when $\Lambda(\phi)$ extends to a bounded mapping*

$$\Lambda(\phi) : \mathcal{S}^\infty(L^2(\lambda_B), L^2(\lambda_C)) \overset{h}{\otimes} \mathcal{S}^\infty(L^2(\lambda_A), L^2(\lambda_B)) \to \mathcal{S}^\infty(L^2(\lambda_A), L^2(\lambda_C)),$$

then the factorization in (iii) holds true. As we saw, this factorization implies the complete boundedness of $\Gamma^{A,B,C}$. Hence, the boundedness of triple operator integrals on $\mathcal{S}^\infty(\mathcal{H})\overset{h}{\otimes}\mathcal{S}^\infty(\mathcal{H})$ implies its complete boundedness.

4.5 Perspectives

Similarly to Section 3.4, one can state several questions concerning multiple operator integrals by changing the spaces $\mathcal{S}^2$ or $\mathcal{S}^1$ by other Schatten classes. First, it would be interesting to have a general definition of multiple operator integrals from $\mathcal{S}^{p_1} \times \ldots \times \mathcal{S}^{p_n}$ into $\mathcal{S}^r$ where $\frac{1}{p_1} + \ldots + \frac{1}{p_n} = 1 - \frac{1}{r}$. Peller gave such definition when the element $\phi \in L^\infty$ belongs to the integral projective tensor product of $L^\infty-$spaces (see [Pel06] or [Pel16]). Then, one can try to obtain necessary or sufficient conditions on ϕ for an element $\Gamma^{A,B,C}(\phi)$ to map for instance $\mathcal{S}^p \times \mathcal{S}^q$ into $\mathcal{S}^1$ (where p and q are conjugate exponents), or for an element $\Gamma^{A_1,A_2,\ldots,A_n}(\phi)$ to map $\mathcal{S}^2 \times \ldots \times \mathcal{S}^2$ into $\mathcal{S}^1$.

Chapter 5

Resolution of Peller's problems

5.1 Statement of the problems

Let $\mathcal{H}$ be a separable complex Hilbert space. In 1953, M. G. Krein [Kre53b] showed that for a self-adjoint (not necessarily bounded) operator A and a self-adjoint operator $K \in \mathcal{S}^1(\mathcal{H})$ there exists a unique function $\xi \in L^1(\mathbb{R})$ such that

$$\mathrm{Tr}(f(A+K) - f(A)) = \int_{\mathbb{R}} f'(t)\xi(t)dt, \tag{5.1}$$

whenever f is from the Wiener class W_1, that is f is a function on $\mathbb{R}$ with Fourier transform of f' in $L^1(\mathbb{R})$.

The function ξ above is called Lifshits-Krein spectral shift function and was firstly introduced in a special case by I. M. Lifshits [Lif52]. It plays an important role in Mathematical Physics and in Scattering Theory, where it appears in the formula of the determinant of a scattering matrix (for detailed discussion we refer to [BY92] and references therein).

Observe that the right-hand side of (5.1) makes sense for every Lipschitz function f. In 1964, M. G. Krein conjectured that the left-hand side of (5.1) also makes sense for every Lipschitz function f. More precisely, Krein's conjecture was the following.

Krein's Conjecture. *For any self-adjoint (not necessarily bounded) operator A, for any self-adjoint operator $K \in \mathcal{S}^1(\mathcal{H})$ and for any Lipschitz function f,*

$$f(A+K) - f(A) \in \mathcal{S}^1. \tag{5.2}$$

The best result concerning the description of the class of functions for which (5.2) holds is due to V. Peller in [Pel85], who established that (5.2) holds for f belonging to the Besov class $B^1_{\infty 1}$ (for a definition of this class, see [Pel85] and references therein). However (5.2) does not hold even for the absolute value function, which is obviously the simplest example of a Lipschitz function (see e.g. [Dav88], [DDPS97]). Moreover, there is an example of a continuously differentiable Lipschitz function f and (bounded) self-adjoint operators A, K with $K \in \mathcal{S}^1$ such that (5.2) does not hold. The first such example is due to Yu. B. Farforovskaya [Far72].

Assume now that K is a self-adjoint operator from the Hilbert-Schmidt class $\mathcal{S}^2$. In 1984, L. S. Koplienko, [Kop84], considered the operator

$$f(A+K) - f(A) - \frac{d}{dt}\Big(f(A+tK)\Big)\Big|_{t=0}, \tag{5.3}$$

where by $\frac{d}{dt}\Big(f(A+tK)\Big)\Big|_{t=0}$ we denote the derivative of the map $t \mapsto f(A+tK) - f(A)$ in the Hilbert-Schmidt norm. He proved that for every fixed self-adjoint operator A there exists a unique function $\eta \in L^1(\mathbb{R})$ such that

$$\mathrm{Tr}\Big(f(A+K) - f(A) - \frac{d}{dt}\Big(f(A+tK)\Big)\Big|_{t=0}\Big) = \int_{\mathbb{R}} f''(t)\eta(t)dt, \tag{5.4}$$

if f is an arbitrary rational function, with poles off $\mathbb{R}$ and $f_{|\mathbb{R}}$ bounded.

The function η is called Koplienko's spectral shift function (for more information about Koplienko's spectral shift function we refer to [GPS08] and references therein).

It is clear that the right-hand side of (5.4) makes sense when f is a twice differentiable function with a bounded second derivative. The natural question is then to describe the class of all these functions f such that the left-hand side of (5.4) is well-defined. Namely, for which function f does the operator (5.3) belong to $\mathcal{S}^1$? The best result to date is again due to V. Peller [Pel05], who established an affirmative answer under the assumption that f belongs to the Besov class $B^2_{\infty 1}$. In the same paper [Pel05], V. Peller stated the following problem.

Peller's problem. [Pel05, Problem 2] *Suppose that f is a twice continuously differentiable function with a bounded second derivative. Let A be a self-adjoint (possibly unbounded) operator and let K be a self-adjoint operator from $\mathcal{S}^2$. Is it true that*

$$f(A+K) - f(A) - \frac{d}{dt}\Big(f(A+tK)\Big)\Big|_{t=0} \in \mathcal{S}^1? \tag{5.5}$$

Now let f be a function on $\mathbb{T}$, admitting a decomposition $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, $z \in \mathbb{T}$ with $\sum_{n=-\infty}^{\infty} |nc_n| < \infty$. Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator and let $Z \in \mathcal{S}^1(\mathcal{H})$ be a self-adjoint operator. Like in the selfadjoint case, M. G. Krein proved a result (see [Kre53a, Theorem 2]) implying that there exists a Lifshits-Krein spectral shift function $\eta \in L^1(\mathbb{T})$ (not depending on f) such that

$$\mathrm{Tr}\big(f(e^{iZ}U) - f(U)\big) = \int_{\mathbb{T}} f'(z)\eta(z)dz. \tag{5.6}$$

Observe that the right-hand side of (5.6) makes sense for every Lipschitz function f. Like in the selfadjoint case, the left-hand side do not always make sense (see [Pel85] or [Far72]), but it does when $f \in B^1_{\infty 1}$ (see [Pel85]).

Let $f \in C^2(\mathbb{T})$, let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator and let $Z \in \mathcal{S}^2(\mathcal{H})$ be a self-adjoint operator. Then the difference operator $f(e^{iZ}U) - f(U)$ belongs to $\mathcal{S}^2(\mathcal{H})$ and the function $t \mapsto f(e^{itZ}U) - f(U)$ from $\mathbb{R}$ into $\mathcal{S}^2(\mathcal{H})$ is differentiable, see e.g. [Pel05, (2.7)]. Let $\frac{d}{dt}\big(f(e^{itZ}U)\big)_{|t=0}$ denote its derivative at $t = 0$. In [Pel05, Problem 1], in connection with the validity of the so-called Koplienko-Neidhardt trace formula, V. V.

Peller asked whether the operator

$$f(e^{iZ}U) - f(U) - \frac{d}{dt}\big(f(e^{itZ}U)\big)_{|t=0} \tag{5.7}$$

necessarily belongs to $\mathcal{S}^1(\mathcal{H})$ under these assumptions. He proved that this holds true whenever f belongs to the Besov class $B^2_{\infty 1}$ and derived a Koplienko-Neidhardt trace formula in this case.

The aim of this chapter is to give a counterexample for both questions (5.5) and (5.7).

Note that in (5.7), the preceding discussion implies that $f(e^{iZ}U)-f(U)-\frac{d}{dt}\big(f(e^{itZ}U)\big)_{|t=0}$ is a well-defined element of $\mathcal{S}^2(\mathcal{H})$. In [Pel05, Theorem 4.6], the author defined the operator in (5.3) for all $f \in B^2_{\infty 1}$ via an approximation process. The precise meaning of (5.3) in the case of an arbitrary self-adjoint operator A and an arbitrary twice continuously differentiable function f is not clear. To give a precise statement to Peller's problem in that case, we first need to study the differentiability of mappings of the form

$$\varphi : t \in \mathbb{R} \mapsto f(A+tK) - f(A) \in \mathcal{S}^2(\mathcal{H})$$

where A and K are selfadjoint, $K \in \mathcal{S}^2(\mathcal{H})$ and f is a n-times differentiable function on $\mathbb{R}$. We will see in Theorem 5.1 that under suitable assumptions on A or f, the map φ will be differentiable and the operator (5.3) will appear as a Taylor formula of second order for ϕ using triple operator integrals. In this case, the operator will be a well-defined element of $\mathcal{S}^2(\mathcal{H})$. In our construction of a counterexample for Peller's problem in the selfadjoint case, the operator A that we obtained is not bounded and the function f does not have a bounded derivative, so that we cannot apply directly Theorem 5.1. However, we will construct A as a direct sum of bounded operators and in that case, the meaning of (5.3) will be unambiguous. We explain this fact in Subsection 5.2.3.

Section 5.2 is dedicated to the connection between perturbation theory for selfadjoint operators and multiple operator integrals and Sections 5.3 and 5.4 concern the construction of counterexamples for Peller's problems in the selfadjoint and the unitary case, respectively.

5.2 Perturbation theory for selfadjoint operators

We recall the definitions of divided differences. For any integer $m \geq 1$, we let $C(\mathbb{R}^m)$ be the vector space of all continuous functions from $\mathbb{R}^m$ into $\mathbb{C}$, we let $C_b(\mathbb{R}^m)$ be the subspace of all bounded continuous functions, and we let $C_0(\mathbb{R}^m)$ be the subspace of all continuous functions vanishing at infinity. Further for any integer $p \geq 1$ we let $C^p(\mathbb{R}^m)$ be the space of all p-times differentiable functions from $\mathbb{R}^m$ into $\mathbb{C}$. Let $f \in C^1(\mathbb{R})$. The divided difference of the first order $f^{[1]} : \mathbb{R}^2 \to \mathbb{C}$ is defined by

$$f^{[1]}(x_0, x_1) := \begin{cases} \frac{f(x_0)-f(x_1)}{x_0-x_1}, & \text{if } x_0 \neq x_1 \\ f'(x_0) & \text{if } x_0 = x_1 \end{cases}, \qquad x_0, x_1 \in \mathbb{R}.$$

The function $f^{[1]}$ belongs to $C(\mathbb{R}^2)$ and if f' is bounded, then $f^{[1]} \in C_b(\mathbb{R}^2)$.

Let $n \geq 2$ and $f \in C^n(\mathbb{R})$. The divided difference of n-th order $f^{[n]} \colon \mathbb{R}^{n+1} \to \mathbb{C}$ is defined recursively by

$$f^{[n]}(x_0, x_1, \ldots, x_n) := \begin{cases} \frac{f^{[n-1]}(x_0,x_2,\ldots,x_n) - f^{[n-1]}(x_1,x_2\ldots,x_n)}{x_0 - x_1}, & \text{if } x_0 \neq x_1 \\ \partial_1 f^{[n-1]}(x_1, x_2, \ldots, x_n) & \text{if } x_0 = x_1 \end{cases},$$

for all $x_0, \ldots, x_n \in \mathbb{R}$.

Here ∂_1 stands for the partial derivation with respect to the first variable. It is well-known that $f^{[n]}$ is symmetric. Therefore, for all $1 \leq i \leq n$ and for all $x_0, \ldots, x_n \in \mathbb{R}$,

$$f^{[n]}(x_0, x_1, \ldots, x_n) = \begin{cases} \frac{f^{[n-1]}(x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n) - f^{[n-1]}(x_0,\ldots,x_{i-2},x_i,x_{i+1},\ldots,x_n)}{x_{i-1} - x_i}, & \text{if } x_{i-1} \neq x_i \\ \partial_i f^{[n-1]}(x_1, \ldots, x_n) & \text{if } x_{i-1} = x_i \end{cases},$$

where ∂_i stands for the partial derivation with respect to the i-th variable.

Note for further use that for all $1 \leq i \leq n$ and for all $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$,

$$f^{[n]}(x_0, \ldots, x_n) = \int_0^1 \partial_i f^{[n-1]}(x_0, \ldots, x_{i-2}, tx_{i-1} + (1-t)x_i, x_{i+1}, \ldots, x_n)\, \mathrm{d}t\,. \tag{5.8}$$

The function $f^{[n]}$ belongs to $C(\mathbb{R}^{n+1})$ and if $f^{(n)}$ is bounded, then $f^{[n]} \in C_b(\mathbb{R}^{n+1})$.

Let A, K be selfadjoint operators on a separable Hilbert space $\mathcal{H}$, and assume that $K \in \mathcal{S}^2(\mathcal{H})$. Let $f \in C^1(\mathbb{R})$. If either f' is bounded or A is bounded, then the restriction of $f^{[1]}$ to $\sigma(A+K) \times \sigma(A)$ is bounded, and hence it makes sense to define the double operator integral mapping $\Gamma^{A+K,A}(f^{[1]}) \colon \mathcal{S}^2(\mathcal{H}) \to \mathcal{S}^2(\mathcal{H})$. One of the early results from double operator integrals theory is that in this case,

$$f(A+K) - f(A) = \left[\Gamma^{A+K,A}(f^{[1]})\right](K). \tag{5.9}$$

See e.g. [PSW02, Theorem 7.4] for a proof of this result. Moreover $t^{-1}\big(f(A+tK) - f(A)\big)$ admits a limit in $\mathcal{S}^2(\mathcal{H})$ when $t \to 0$ and denoting this limit by $\frac{d}{dt} f(A+tK)_{|t=0}$, we have

$$\frac{d}{dt} f(A+tK)_{|t=0} = \left[\Gamma^{A,A}(f^{[1]})\right](K). \tag{5.10}$$

A proof of that result will be given in Theorem 5.1.

The main result of this section is the existence of higher order derivatives in the $\mathcal{S}^2$-norm and an analog of (5.9) for the higher order perturbation operator

$$f(A+K) - f(A) - \sum_{k=1}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k} f(A+tK)_{|t=0}.$$

For any integer $p \geq 1$, we denote by $\mathcal{D}^p(\mathbb{R})$ the space of p-times differentiable functions

$$\phi : \mathbb{R} \to \mathcal{S}^2(\mathcal{H})$$

and we denote by $\phi^{(p)} : \mathbb{R} \to \mathcal{S}^2(\mathcal{H})$ the $p-$th derivative of ϕ.

We have the following result:

Theorem 5.1. *Let A and K be selfadjoint operators on a separable Hilbert space $\mathcal{H}$ with $K \in \mathcal{S}^2(\mathcal{H})$. Let $n \geq 1$ and $f \in C^n(\mathbb{R})$. Assume either that A is bounded or that for all $1 \leq i \leq n$, $f^{(i)}$ is bounded. Then, one may define*

$$\varphi : t \in \mathbb{R} \mapsto f(A+tK) - f(A) \in \mathcal{S}^2(\mathcal{H}).$$

(i) *The function φ belongs to $\mathcal{D}^n(\mathbb{R})$ and for any integer $1 \leq k \leq n$ and any $t \in \mathbb{R}$,*

$$\frac{1}{k!}\varphi^{(k)}(t) = \left[\Gamma^{A+tK,A+tK,\ldots,A+tK}(f^{[k]})\right](K,\ldots,K). \tag{5.11}$$

(ii) *We have*

$$f(A+K) - f(A) - \sum_{k=1}^{n-1} \frac{1}{k!}\varphi^{(k)}(0) = \left[\Gamma^{A+K,A,\ldots,A}(f^{[n]})\right](K,\ldots,K). \tag{5.12}$$

This theorem will be proved in Subsection 5.2.2.

5.2.1 Approximation in multiple operator integrals

In this section, we will extend to the setting of multiple operator integrals a result of [PS04] concerning an approximation property for double operator integrals. Following the latter reference we will use resolvent strong convergence. Let A be a selfadjoint operator on $\mathcal{H}$. We say that a sequence $(A_j)_j$ of selfadjoint operators is resolvent strongly convergent to A if for any $z \in \mathbb{C} \setminus \mathbb{R}$, $(z-A_j)^{-1} \to (z-A)^{-1}$ in the strong operator topology (SOT). According to [RS80, Theorem 8.20], this is equivalent to

$$\forall\, f \in C_b(\mathbb{R}), \qquad f(A_j) \overset{SOT}{\longrightarrow} f(A) \qquad \text{when } j \to \infty. \tag{5.13}$$

The following lemma states that any selfadjoint operator is the limit (in the above sense) of bounded selfadjoint operators.

Lemma 5.2. *Let A be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$. Let E be the spectral measure of A and define $A_n := E((-n,n))A$ for every $n \in \mathbb{N}$. Then, the sequence of bounded self-adjoint operators $\{A_n\}_{n=1}^{\infty}$ converges to A in the strong resolvent sense.*

Proof. Since $E((-n,n))$ converges to I in the strong operator topology,

$$\lim_{n\to\infty} A_n g = Ag \tag{5.14}$$

for every $g \in D$, where D is the domain of A. Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $f \in \mathcal{H}$. The mapping $A - z : D \to \mathcal{H}$ is a bijection so that $(A-z)^{-1}f \in D$. By standard properties of the resolvent,

$$(A_n - z)^{-1}f - (A-z)^{-1}f = (A_n - z)^{-1}(A - A_n)(A-z)^{-1}f. \tag{5.15}$$

The result follows from combining (5.14) and (5.15) and applying uniform boundedness of $(A_n - z)^{-1}$. $\square$

If $A_j \to A$ and $B_j \to B$ resolvent strongly, then [PS04, Prop. 3.2] shows that for any $\psi \in C_b(\mathbb{R}^2)$,

$$\Gamma^{A_j,B_j}(\psi) \overset{SOT}{\longrightarrow} \Gamma^{A,B}(\psi) \qquad \text{when } j \to \infty.$$

The following is a multiple operator integral version of this result.

Proposition 5.3. *Let $A_1, \dots, A_n$ be selfadjoint operators on a separable Hilbert space $\mathcal{H}$ and let, for all $1 \le i \le n$, $(A_i^j)_{j\in\mathbb{N}}$ be a sequence of selfadjoint operators on $\mathcal{H}$ resolvent strongly convergent to A_i. Then for any $\phi \in C_b(\mathbb{R}^n)$ and for any $K_1, \dots, K_{n-1} \in \mathcal{S}^2(\mathcal{H})$,*

$$\lim_{j\to+\infty} \left\| \Gamma^{A_1^j,\dots,A_n^j}(\phi)(K_1,\dots,K_{n-1}) - \Gamma^{A_1,\dots,A_n}(\phi)(K_1,\dots,K_{n-1}) \right\|_2 = 0. \tag{5.16}$$

Proof. For simplicity we write $\Gamma = \Gamma^{A_1,\dots,A_n}$ and $\Gamma_j = \Gamma^{A_1^j,\dots,A_n^j}$ along this proof. Since $\overline{\mathcal{H}} \otimes \mathcal{H}$ is dense in $\mathcal{S}^2(\mathcal{H})$ and $\|\Gamma_j\| = 1$ for any $j \ge 1$, it suffices to prove (5.16) in the case when $K_1, \dots, K_{n-1}$ are elementary tensors. Thus from now on we assume that for all $1 \le i \le n-1$,

$$K_i = \overline{h_i} \otimes h_i'$$

with $h_i, h_i' \in \mathcal{H}$.

Assume first that $\phi = u_1 \otimes \cdots \otimes u_n$, with $u_i \in C_b(\mathbb{R})$ for all i. In this case,

$$\begin{aligned}\Gamma_j(\phi)(K_1,\dots,K_{n-1}) &= u_1(A_1^j)(\overline{h_1} \otimes h_1') \dots (\overline{h_{n-1}} \otimes h_{n-1}') u_n(A_n^j) \\ &= \left(\prod_{k=2}^{n-1} \langle u_k(A_k^j) h_k', h_{k-1} \rangle \right) \overline{\overline{u_n}(A_n^j)(h_{n-1})} \otimes u_1(A_1^j)(h_1').\end{aligned}$$

By the assumption and (5.13), this converges to

$$\left(\prod_{k=2}^{n-1} \langle u_k(A_k) h_k', h_{k-1} \rangle \right) \overline{\overline{u_n}(A_n)(h_{n-1})} \otimes u_1(A_1)(h_1'),$$

which in turn is equal to $\Gamma(\phi)(K_1, \dots, K_{n-1})$. This shows (5.16) in this special case. By linearity and standard approximation, this implies that (5.16) holds true whenever ϕ belongs to the uniform closure of $C_b(\mathbb{R}) \otimes \cdots \otimes C_b(\mathbb{R})$. In particular, (5.16) holds true when $\phi \in C_0(\mathbb{R}^n)$.

The rest of the proof consists in reducing to this case by a more subtle (i.e. non uniform) approximation process. Let $(g_k)_{k\ge1}$ be a sequence of functions in $C_0(\mathbb{R})$ satisfying the following two properties:

$$\forall\, k \ge 1, \quad 0 \le g_k \le 1, \qquad \text{and} \qquad \forall\, r \in \mathbb{R}, \quad g_k(r) \overset{k\to\infty}{\longrightarrow} 1.$$

These properties imply that for all $1 \le i \le n$, $g_k(A_i) \to I_{\mathcal{H}}$ strongly. Indeed let $h \in \mathcal{H}$, then by the Spectral theorem,

$$\|g_k(A_i)h - h\|^2 = \int_{\sigma(A_i)} (1 - g_k(r))^2 \, \mathrm{d}E_{h,h}^{A_i}(r).$$

Then by Lebesgue's dominated convergence theorem, $\|g_k(A_i)h - h\|^2 \to 0$ when $k \to \infty$.

We consider an arbitrary $\phi \in C_b(\mathbb{R}^n)$ and set

$$\phi_k = (g_k \otimes g_k^2 \otimes \cdots \otimes g_k^2 \otimes g_k)\phi, \qquad k \geq 1.$$

Clearly each ϕ_k belongs to $C_0(\mathbb{R}^n)$, hence satisfies (5.16). A crucial observation is that for all $j, k \geq 1$,

$$\Gamma_j(\phi_k)(K_1, \ldots, K_{n-1}) = \Gamma_j(\phi)\left(g_k(A_1^j)K_1 g_k(A_2^j), \ldots, g_k(A_{n-1}^j)K_{n-1}g_k(A_n^j)\right). \tag{5.17}$$

The argument for this identity is essentially the same as the one for the proof of 4.9. One first checks the validity of (5.17) in the case when ϕ belongs to $C_b(\mathbb{R}) \otimes \cdots \otimes C_b(\mathbb{R})$, then one uses the w^*-continuity of Γ_j to obtain the general case. Details are left to the reader. Likewise we have, for all $k \geq 1$,

$$\Gamma(\phi_k)(K_1, \ldots, K_{n-1}) = \Gamma(\phi)\left(g_k(A_1)K_1 g_k(A_2), \ldots, g_k(A_{n-1})K_{n-1}g_k(A_n)\right). \tag{5.18}$$

For any $k \geq 1$ and any $1 \leq i \leq n-1$,

$$g_k(A_i)K_i g_k(A_{i+1}) = g_k(A_i)(\overline{h_i} \otimes h_i')g_k(A_{i+1}) = \overline{g_k(A_{i+1})(h_i)} \otimes g_k(A_i)(h_i'),$$

hence $g_k(A_i)K_i g_k(A_{i+1}) \to K_i$ in $\mathcal{S}^2(\mathcal{H})$ when $k \to \infty$.

Let $\varepsilon > 0$. According to the above observation, we fix $k_0 \geq 1$ such that for any $1 \leq i \leq n-1$,

$$\|g_{k_0}(A_i)K_i g_{k_0}(A_{i+1}) - K_i\|_2 \leq \varepsilon.$$

Hence, there exists a constant $\alpha > 0$ such that

$$\|\Gamma(\phi_{k_0})(K_1, \ldots, K_{n-1}) - \Gamma(\phi)(K_1, \ldots, K_{n-1})\|_2 \leq \alpha\varepsilon.$$

Now, using that for any $1 \leq i \leq n-1$, $g_{k_0}(A_i^j)K_i g_{k_0}(A_{i+1}^j) = \overline{g_{k_0}(A_{i+1}^j)(h_i)} \otimes g_{k_0}(A_i^j)(h_i')$ and the fact that $g_{k_0}(A_i^j) \to g_{k_0}(A_i)$ and $g_{k_0}(A_{i+1}^j) \to g_{k_0}(A_{i+1})$ strongly when $j \to \infty$, we see that $g_{k_0}(A_i^j)K_i g_{k_0}(A_{i+1}^j) \to g_{k_0}(A_i)K_i g_{k_0}(A_{i+1})$ in $\mathcal{S}^2(\mathcal{H})$ when $j \to \infty$. Hence, for a large enough $j_0 \geq 1$, we have, for any $1 \leq i \leq n-1$,

$$\|g_{k_0}(A_i^j)K_i g_{k_0}(A_{i+1}^j) - K_i\|_2 \leq 2\varepsilon$$

for any $j \geq j_0$. We deduce that there exists a constant $\beta > 0$ such that

$$\forall\, j \geq j_0, \qquad \|\Gamma_j(\phi_{k_0})(K_1, \ldots, K_{n-1}) - \Gamma_j(\phi)(K_1, \ldots, K_{n-1})\|_2 \leq \beta\varepsilon.$$

Now recall that ϕ_k satisfies (5.16). Hence changing j_0 into a bigger integer if necessary we also have

$$\forall\, j \geq j_0, \qquad \|\Gamma_j(\phi_{k_0})(K_1, \ldots, K_{n-1}) - \Gamma(\phi_{k_0})(K_1, \ldots, K_{n-1})\|_2 \leq \varepsilon.$$

We deduce from the above three estimates that

$$\forall\, j \geq j_0, \qquad \|\Gamma_j(\phi)(K_1, \ldots, K_{n-1}) - \Gamma(\phi)(K_1, \ldots, K_{n-1})\|_2 \leq (\alpha + \beta + 1)\varepsilon.$$

This shows that ϕ satisfies (5.16).

□

We finish this section with a lemma that will be used in Section 5.2.2.

Lemma 5.4. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of self-adjoint operators converging to a self-adjoint operator A in the strong resolvent sense. Let K be a bounded self-adjoint operator. Then, $\{A_n + K\}_{n=1}^{\infty}$ converges in the strong resolvent sense to $A + K$.*

Proof. Let $z \in \mathbb{C}$ be such that $\mathrm{Im}\,(z) \neq 0$. Note that

$$(A - z)(A + K - z)^{-1} = I - K(A + K - z)^{-1} \tag{5.19}$$

and, on the domain of A_n,

$$(A_n + K - z)^{-1}(A_n - z) = I - (A_n + K - z)^{-1}K. \tag{5.20}$$

These operators are bounded in the operator norm by $1 + ||K||/|\mathrm{Im}\,(z)|$. By simple algebraic manipulations,

$$\begin{aligned}&(A_n + K - z)^{-1} - (A + K - z)^{-1}\\&= (A_n + K - z)^{-1}\big(I - K(A + K - z)^{-1}\big) - \big(I - (A_n + K - z)^{-1}K\big)(A + K - z)^{-1}.\end{aligned}$$

Combining the latter with (5.19) and (5.20) gives

$$\begin{aligned}&(A_n + K - z)^{-1} - (A + K - z)^{-1}\\&= (A_n + K - z)^{-1}(A - z)(A + K - z)^{-1} - (A_n + K - z)^{-1}(A_n - z)(A + K - z)^{-1}\\&= (A_n + K - z)^{-1}(A_n - z)\big((A_n - z)^{-1} - (A - z)^{-1}\big)(A - z)(A + K - z)^{-1}.\end{aligned} \tag{5.21}$$

Let $f \in \mathcal{H}$. It follows from (5.21) that

$$\begin{aligned}&\big\|\big((A_n + K - z)^{-1} - (A + K - z)^{-1}\big)f\big\|\\&\leq \big(1 + ||K||/|\mathrm{Im}\,(z)|\big)\,\big\|\big((A_n - z)^{-1} - (A - z)^{-1}\big)\big((A - z)(A + K - z)^{-1}f\big)\big\|,\end{aligned}$$

completing the proof of the lemma. □

5.2.2 Proof of the main result

In this section, we will prove Theorem 5.1. First, we will need the following lemmas and corollary.

Lemma 5.5. *Let $n \geq 3$ and $1 \leq k \leq n - 2$. Let $u \in C_b(\mathbb{R}^{k+1})$ and $v \in C_b(\mathbb{R}^{n-k})$. We set, for any $(t_1, \ldots, t_n) \in \mathbb{R}^n$,*

$$(uv)(t_1, \ldots, t_n) = u(t_1, \ldots, t_{k+1})v(t_{k+1}, \ldots, t_n).$$

Then, $uv \in C_b(\mathbb{R}^n)$.
Let $A_1, \ldots, A_n$ be selfadjoint operators on a separable Hilbert space $\mathcal{H}$. Then, for any $K_1, \ldots, K_{n-1} \in$

$\mathcal{S}^2(\mathcal{H})$,

$$\begin{aligned}&\Gamma^{A_1,\ldots,A_n}(uv)(K_1,\ldots,K_{n-1})\\&\quad=\Gamma^{A_1,\ldots,A_{k+1}}(u)(K_1,\ldots,K_k)\Gamma^{A_{k+1},\ldots,A_n}(v)(K_{k+1},\ldots,K_{n-1}).\end{aligned}$$

Proof. We first prove the formula when u and v are elementary tensors of elements of $C_b(\mathbb{R})$. Then, one uses the w^*-continuity of multiple operator integrals like in Lemma 4.9 to obtain the general case. □

Lemma 5.6. *Let $n \geq 2$ be an integer. Let $A_1, \ldots, A_{n-1}, A, B$ be bounded selfadjoint operators on a separable Hilbert space $\mathcal{H}$ and assume that $B - A \in \mathcal{S}^2(\mathcal{H})$. Let $f \in C^n(\mathbb{R})$. Then, for all $K_1, \ldots, K_{n-1} \in \mathcal{S}^2(\mathcal{H})$ and for any $1 \leq i \leq n$ we have*

$$\begin{aligned}&\left[\Gamma^{A_1,\ldots A_{i-1},B,A_i,\ldots,A_{n-1}}(f^{[n-1]})\right](K_1,\ldots,K_{n-1})\\&\quad-\left[\Gamma^{A_1,\ldots A_{i-1},A,A_i,\ldots,A_{n-1}}(f^{[n-1]})\right](K_1,\ldots,K_{n-1})\\&=\left[\Gamma^{A_1,\ldots,A_{i-1},B,A,A_i,\ldots,A_{n-1}}(f^{[n]})\right](K_1,\ldots,K_{i-1},B-A,K_i,\ldots K_{n-1}).\end{aligned}$$

Proof. It will be convenient to extend the definition of the divided difference as follows. Let $m \in \mathbb{N}^*$ and $1 \leq i \leq m$. For any $\phi \in C^1(\mathbb{R}^m)$, we define a function $\phi_i^{[1]} \colon \mathbb{R}^{m+1} \to \mathbb{C}$ by

$$\phi_i^{[1]}(x_0,\ldots,x_m)=\int_0^1 \partial_i\phi(x_0,\ldots,x_{i-2},tx_{i-1}+(1-t)x_i,x_{i+1},\ldots,x_m)\,\mathrm{d}t$$

for all $(x_0, \ldots, x_m) \in \mathbb{R}^{m+1}$. The index i in the notation $\phi_i^{[1]}$ refers to the i-th variable derivation ∂_i. It follows from (5.8) that for any $f \in C^n(\mathbb{R})$,

$$(f^{[n-1]})_i^{[1]} = f^{[n]}. \tag{5.22}$$

For $\phi \in C(\mathbb{R}^n)$, write

$$\Gamma_A(\phi)=\left[\Gamma^{A_1,\ldots A_{i-1},A,A_i,\ldots,A_{n-1}}(\phi)\right](K_1,\ldots,K_{n-1})$$

and

$$\Gamma_B(\phi)=\left[\Gamma^{A_1,\ldots A_{i-1},B,A_i,\ldots,A_{n-1}}(\phi)\right](K_1,\ldots,K_{n-1}).$$

For $\psi \in C(\mathbb{R}^{n+1})$, write

$$\Gamma_{B,A}(\psi)=\left[\Gamma^{A_1,\ldots,A_{i-1},B,A,A_i,\ldots,A_{n-1}}(\psi)\right](K_1,\ldots,K_{i-1},B-A,K_i,\ldots,K_{n-1}).$$

We will show that for any $\phi \in C^1(\mathbb{R}^n)$,

$$\Gamma_B(\phi)-\Gamma_A(\phi)=\Gamma_{B,A}\left(\phi_i^{[1]}\right). \tag{5.23}$$

Then the result follows by applying this formula to $\phi = f^{[n-1]}$, together with (5.22).

Assume first that $\phi = u_1 \otimes \cdots \otimes u_n$ for functions $u_j \in C^1(\mathbb{R})$, i.e $\phi(t_1, \ldots, t_n) = u_1(t_1) \ldots u_n(t_n)$ for any $(t_1, \ldots, t_n) \in \mathbb{R}^n$. Then

$$\partial_i \phi = u_1 \otimes \cdots \otimes u_{i-1} \otimes u_i' \otimes u_{i+1} \otimes \cdots \otimes u_n.$$

Hence,

$$\phi_i^{[1]} = u_1 \otimes \cdots \otimes u_{i-1} \otimes u_i^{[1]} \otimes u_{i+1} \otimes \cdots \otimes u_n.$$

By Lemma 5.5 we have

$$\begin{aligned}
&\Gamma_{B,A}(\phi_i^{[1]}) \\
&= \left[\Gamma^{A_1,\ldots,A_{i-1},B}(u_1 \otimes \cdots \otimes u_{i-1} \otimes 1)\right](K_1, \ldots, K_{i-1}) \left[\Gamma^{B,A,A_i}(u_i^{[1]} \otimes 1)\right](B - A, K_i) \\
&\quad \left[\Gamma^{A_i,\ldots,A_{n-1}}(u_{i+1} \otimes \cdots \otimes u_n)\right](K_{i+1}, \ldots, K_{n-1}).
\end{aligned}$$

We have, by (5.9),

$$\begin{aligned}
\left[\Gamma^{B,A,A_i}(u_i^{[1]} \otimes 1)\right](B - A, K_i) &= \left[\Gamma^{B,A}(u_i^{[1]})\right](B - A)K_i \\
&= (u_i(B) - u_i(A))K_i.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Gamma_{B,A}(\phi_i^{[1]}) \\
&= u_1(A_1)K_1 \ldots u_{i-1}(A_{i-1})K_{i-1}(u_i(B) - u_i(A))K_i u_{i+1}(A_i)K_{i+1} \ldots u_n(A_{n-1}) \\
&= u_1(A_1)K_1 \ldots u_{i-1}(A_{i-1})K_{i-1}u_i(B)K_i u_{i+1}(A_i)K_{i+1} \ldots u_n(A_{n-1}) \\
&\quad - u_1(A_1)K_1 \ldots u_{i-1}(A_{i-1})K_{i-1}u_i(A)K_i u_{i+1}(A_i)K_{i+1} \ldots u_n(A_{n-1}) \\
&= \Gamma_B(\phi) - \Gamma_A(\phi).
\end{aligned}$$

This shows (5.23) in the case when $\phi = u_1 \otimes \cdots \otimes u_n$. By linearity this immediately implies that (5.23) holds true whenever $\phi \in C^1(\mathbb{R}) \otimes \cdots \otimes C^1(\mathbb{R})$. Note that this space contains the $n-$variable polynomial functions.

Now consider an arbitrary $\phi \in C^1(\mathbb{R}^n)$. Let $M > 0$ be a constant such that the spectra of $A_1, \ldots, A_{n-1}, A$ and B are included in $[-M, M]$. By continuity of $\partial_i \phi$ there exists a sequence $(Q_m)_{m \geq 1}$ of $n-$variable polynomial functions such that $Q_m \to \partial_i \phi$ uniformly on $[-M, M]^n$. For any $m \geq 1$, we set

$$P_m(t_1, \ldots, t_n) = \int_0^{t_i} Q_m(t_1, \ldots, t_{i-1}, \theta, t_{i+1}, \ldots, t_n)\, \mathrm{d}\theta$$

for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$. This is also an n-variable polynomial function. Next we introduce $w(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) = \phi(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)$. w belongs to $C^1(\mathbb{R}^{n-1})$ and for any real numbers $t_1, \ldots, t_n$, we have

$$\phi(t_1, \ldots, t_n) = w(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) + \int_0^{t_i} \partial_i \phi(t_1, \ldots, t_{i-1}, \theta, t_{i+1}, \ldots, t_n)\, \mathrm{d}\theta.$$

Hence,

$$
\begin{aligned}
&|\phi(t_1,\dots,t_n) - w(t_1,\dots,t_{i-1},t_{i+1},\dots,t_n) - P_n(t_1,\dots,t_n)| \\
&\le \int_0^{t_i} |\partial_i\phi(t_1,\dots,t_{i-1},\theta,t_{i+1},\dots,t_n) - Q_m(t_1,\dots,t_{i-1},\theta,t_{i+1},\dots,t_n)|\,\mathrm{d}\theta.
\end{aligned}
$$

Consequently, $P_m + w \to \phi$ uniformly on $[-M,M]^n$. Let $(w_m)_{m\in\mathbb{N}}$ be a sequence of $(n-1)-$variable polynomial functions converging uniformly to w on $[-M,M]^{n-1}$. The latter implies that $P_m + w_m \to \phi$ uniformly on $[-M,M]^n$. By construction, $\partial_i P_m = Q_m$ and $\partial_i w_m = 0$ hence we also obtain that $(P_m + w_m)_i^{[1]} \to \phi_i^{[1]}$ uniformly on $[-M,M]^{n+1}$. Since $P_m + w_m$ belongs to $C^1(\mathbb{R})\otimes\cdots\otimes C^1(\mathbb{R})$, it satisfies (5.23). The above approximation property implies that ϕ satisfies (5.23) as well. □

Corollary 5.7. *Let $n \ge 2$ be an integer. Let $A_1,\dots,A_{n-1},A,K$ be selfadjoint operators on a separable Hilbert space $\mathcal{H}$ and assume that $K \in \mathcal{S}^2(\mathcal{H})$. Let $f \in C^n(\mathbb{R})$ be such that $f^{(n-1)}$ and $f^{(n)}$ are bounded. Then, for all $K_1,\dots,K_{n-1} \in \mathcal{S}^2(\mathcal{H})$ and for any $1 \le i \le n$ we have*

$$
\begin{aligned}
&\left[\Gamma^{A_1,\dots A_{i-1},A+K,A_i,\dots,A_{n-1}}(f^{[n-1]})\right](K_1,\dots,K_{n-1}) \\
&\quad - \left[\Gamma^{A_1,\dots A_{i-1},A,A_i,\dots,A_{n-1}}(f^{[n-1]})\right](K_1,\dots,K_{n-1}) \\
&= \left[\Gamma^{A_1,\dots,A_{i-1},A+K,A,A_i,\dots,A_{n-1}}(f^{[n]})\right](K_1,\dots,K_{i-1},K,K_i,\dots K_{n-1}).
\end{aligned}
$$

Proof. For all $1 \le k \le n-1$, let $(A_k^j)_{j\in\mathbb{N}}$ be a sequence of bounded selfadjoint operators on $\mathcal{H}$ converging resolvent strongly to A_k. Such sequence exists by Lemma 5.2. Similarly, let $(A^j)_{j\in\mathbb{N}}$ be a sequence of bounded selfadjoint operators converging resolvent strongly to A. According to Lemma 5.6, we have, for all j,

$$
\begin{aligned}
&\left[\Gamma^{A_1^j,\dots A_{i-1}^j,A^j+K,A_i^j,\dots,A_{n-1}^j}(f^{[n-1]})\right](K_1,\dots,K_{n-1}) \\
&\quad - \left[\Gamma^{A_1^j,\dots A_{i-1}^j,A^j,A_i^j,\dots,A_{n-1}^j}(f^{[n-1]})\right](K_1,\dots,K_{n-1}) \\
&= \left[\Gamma^{A_1^j,\dots,A_{i-1}^j,A^j+K,A^j,A_i^j,\dots,A_{n-1}^j}(f^{[n]})\right](K_1,\dots,K_{i-1},K,K_i,\dots K_{n-1}).
\end{aligned}
$$

By Lemma 5.4, $A^j + K \to A + K$ resolvent strongly when $j \to \infty$. Moreover, the boundedness of $f^{(n-1)}$ and $f^{(n)}$ imply that of $f^{[n-1]}$ and $f^{[n]}$. Hence, we obtain the desired equality by passing to the limit in the above equality thanks to Proposition 5.3. □

Proof of Theorem 5.1. 1. Assume first that A is bounded.
(i) We prove the first point by induction on k, $1 \le k \le n$. Let $k = 1$ and $t \in \mathbb{R}$. We want to show that the limit

$$\lim_{s\to 0} \frac{\varphi(t+s) - \varphi(t)}{s}$$

exists in $\mathcal{S}^2(\mathcal{H})$ and is equal to $\left[\Gamma^{A+tK,A+tK}(f^{[1]})\right](K)$.

By (5.9) we have

$$\begin{aligned}\frac{\varphi(t+s)-\varphi(t)}{s} &= \frac{f(A+(t+s)K)-f(A+tK)}{s} \\ &= \left[\Gamma^{A+(t+s)K,A+tK}(f^{[1]})\right](K).\end{aligned}$$

By Lemma 5.4, we get that $A+(t+s)K \to A+tK$ resolvent strongly as $s \to 0$. By assumption A and K are bounded so there exists a bounded interval $I \subset \mathbb{R}$ such that for s small enough, $\sigma(A+(t+s)K) \subset I$. Since $f \in C^1(\mathbb{R})$, $f^{[1]}$ is continuous hence bounded on $I \times I$. Let $F \in C_b(\mathbb{R}^2)$ be such that $F_{|I\times I} = f^{[1]}$. By Proposition 5.3 applied to F we get

$$\lim_{s\to 0} \left[\Gamma^{A+(t+s)K,A+tK}(f^{[1]})\right](K) = \left[\Gamma^{A+tK,A+tK}(f^{[1]})\right](K) \text{ in } \mathcal{S}^2(\mathcal{H}),$$

which concludes the proof when $k=1$.

Now let $1 \leq k \leq n-1$ and assume that $\varphi \in \mathcal{D}^k(\mathbb{R})$ and for all $1 \leq j \leq k$ and $t \in \mathbb{R}$,

$$\varphi^{(j)}(t) = j! \left[\Gamma^{A+tK,A+tK,\dots,A+tK}(f^{[j]})\right](K,\dots,K). \tag{5.24}$$

We want to prove that $\varphi \in \mathcal{D}^{k+1}(\mathbb{R})$ with a derivative of $(k+1)$-th order given by (5.11). Let $s,t \in \mathbb{R}$. We have

$$\begin{aligned}&\frac{\varphi^{(k)}(t+s)-\varphi^{(k)}(t)}{s} \\ &= \frac{k!}{s}\left[\Gamma^{A+(t+s)K,\dots,A+(t+s)K}(f^{[k]}) - \Gamma^{A+tK,\dots,A+tK}(f^{[k]})\right](K,\dots,K) \\ &= \frac{k!}{s}\sum_{i=1}^{k+1}\left[\Gamma^{(A+tK)^{i-1},(A+(t+s)K)^{k-i+2}}(f^{[k]}) - \Gamma^{(A+tK)^{i},(A+(t+s)K)^{k-i+1}}(f^{[k]})\right](K,\dots,K)\end{aligned}$$

where for instance $(A+tK)^i = A+tK,\dots,A+tK$ (i terms). By Lemma 5.6, we have for all $1 \leq i \leq k+1$,

$$\begin{aligned}&\frac{1}{s}\left[\Gamma^{(A+tK)^{i-1},(A+(t+s)K)^{k-i+2}}(f^{[k]}) - \Gamma^{(A+tK)^{i},(A+(t+s)K)^{k-i+1}}(f^{[k]})\right](K,\dots,K) \\ &= \frac{1}{s}\left[\Gamma^{(A+tK)^{i-1},A+(t+s)K,A+tK,(A+(t+s)K)^{k-i+1}}(f^{[k+1]})\right](K,\dots,K,sK,K,\dots,K) \\ &= \left[\Gamma^{(A+tK)^{i-1},A+(t+s)K,A+tK,(A+(t+s)K)^{k-i+1}}(f^{[k+1]})\right](K,\dots,K).\end{aligned}$$

Moreover, using resolvent convergence like in the first part of the proof, we can see that this term converges in $\mathcal{S}^2(\mathcal{H})$, as s goes to 0, to

$$\left[\Gamma^{A+tK,\dots,A+tK}(f^{[k+1]})\right](K,\dots,K).$$

Hence,

$$\lim_{s\to 0}\frac{\varphi^{(k)}(t+s)-\varphi^{(k)}(t)}{s} = k!\sum_{i=1}^{k+1}\left[\Gamma^{A+tK,\dots,A+tK}(f^{[k+1]})\right](K,\dots,K)$$

$$= (k+1)! \left[\Gamma^{A+tK,\dots,A+tK}(f^{[k+1]})\right](K,\dots,K)$$

which concludes the proof of (i).

(ii). We will prove the second point by induction on n. The case $n = 1$ follows from (5.9). Now let $n \in \mathbb{N}$ and $f \in C^{n+1}(\mathbb{R})$. Assume that we have

$$f(A+K) - f(A) - \sum_{k=1}^{n-1} \frac{1}{k!}\varphi^{(k)}(0) = \left[\Gamma^{A+K,A,\dots,A}(f^{[n]})\right](K,\dots,K).$$

We have

$$\begin{aligned} f(A+K) - f(A) - \sum_{k=1}^{n} \frac{1}{k!}\varphi^{(k)}(0) &= f(A+K) - f(A) - \sum_{k=1}^{n-1} \frac{1}{k!}\varphi^{(k)}(0) - \frac{1}{n!}\varphi^{(n)}(0) \\ &= \left[\Gamma^{A+K,A,\dots,A}(f^{[n]})\right](K,\dots,K) - \frac{1}{n!}\varphi^{(n)}(0). \end{aligned}$$

By the first point of the theorem, we have

$$\frac{1}{n!}\varphi^{(n)}(0) = \left[\Gamma^{A,A,\dots,A}(f^{[n]})\right](K,\dots,K).$$

Using Lemma 5.6, we obtain

$$f(A+K) - f(A) - \sum_{k=1}^{n} \frac{1}{k!}\varphi^{(k)}(0) = \left[\Gamma^{A+K,A,\dots,A}(f^{[n+1]})\right](K,\dots,K)$$

which is the desired equality.

2. Assume now that A is unbounded and that for all $1 \leq i \leq n$, $f^{(i)}$ is bounded. Then, for all $1 \leq i \leq n$, $f^{[i]}$ is bounded. Hence, applying Corollary 5.7 instead of Lemma 5.6 and following the same lines as in the proof of the bounded case, we obtain the unbounded case. □

Theorem 5.1, Proposition 5.3 and Lemma 5.4 have the following consequence.

Corollary 5.8. *Let A be a selfadjoint operator on a separable Hilbert space $\mathcal{H}$ and let $(A_j)_{j\in\mathbb{N}}$ be a sequence of bounded selfadjoint operators on $\mathcal{H}$ converging resolvent strongly to A. Let $n \geq 1$ be an integer and let $f \in C^n(\mathbb{R})$ be such that $f^{(n)}$ is bounded. Let $K = K^* \in \mathcal{S}^2(\mathcal{H})$ and define, for any $j \geq 1$,*

$$\varphi_j : t \in \mathbb{R} \mapsto f(A_j + tK) - f(A_j) \in \mathcal{S}^2(\mathcal{H}).$$

Then, for any $t \in \mathbb{R}$,

$$\lim_{j\to\infty} \frac{\varphi_j^{(n)}(t)}{n!} = \left[\Gamma^{A+tK,\dots,A+tK}(f^{[n]})\right](K,\dots,K)$$

and

$$\lim_{j\to\infty}\left(f(A_j+K)-f(A_j)-\sum_{k=1}^{n-1}\frac{1}{k!}\varphi_j^{(k)}(0)\right)=\left[\Gamma^{A+K,A,\ldots,A}(f^{[n]})\right](K,\ldots,K),$$

where the limits are in $\mathcal{S}^2(\mathcal{H})$.

5.2.3 Connection with Peller's problem

The results obtained in this section will allow us to give a meaning and a concrete approximation process for the operator

$$f(A+K)-f(A)-\frac{d}{dt}\Big(f(A+tK)\Big)\Big|_{t=0}$$

when A and K are selfadjoint operators on a separable Hilbert space $\mathcal{H}$, $K\in\mathcal{S}^2(\mathcal{H})$ and $f\in C^2(\mathbb{R})$ with a bounded second derivative, in the case when $\mathcal{H}$ is a direct sum and A and K are a direct sum of operators. Thus, we will assume that $\mathcal{H}=\overset{2}{\bigoplus}_{i\in\mathbb{N}}\mathcal{H}_i$ is the direct sum of finite dimensional Hilbert spaces $\mathcal{H}_i$ and that A and K are of the form

$$A=\bigoplus_{i=1}^{+\infty}\tilde{A}_i \text{ and } K=\bigoplus_{i=1}^{+\infty}\tilde{K}_i$$

where for all $i\in\mathbb{N}$, A_i and K_i are bounded selfadjoint operators acting on $\mathcal{H}_i$ such that

$$\|K\|_2^2=\sum_{i=1}^{\infty}\|\tilde{K}_i\|_2^2<\infty. \tag{5.25}$$

Set, for $n\geq 1$,

$$A_n=\left(\bigoplus_{i=1}^{n}\tilde{A}_i\right)\oplus\left(\bigoplus_{i=n+1}^{+\infty}0_{\mathcal{H}_i}\right) \text{ and } K_n=\left(\bigoplus_{i=1}^{n}\tilde{K}_i\right)\oplus\left(\bigoplus_{i=n+1}^{+\infty}0_{\mathcal{H}_i}\right).$$

If $h\in C_b(\mathbb{R})$ then

$$h(A)=\bigoplus_{i=1}^{+\infty}h(\tilde{A}_i)$$

and for any $n\geq 1$,

$$h(A_n)=\left(\bigoplus_{i=1}^{n}h(\tilde{A}_i)\right)\oplus\left(\bigoplus_{i=n+1}^{+\infty}h(0)I_{\mathcal{H}_i}\right).$$

Therefore, it is easy to see that $A_n\to A$ resolvent strongly as $n\to+\infty$. Similarly, $K_n\to K$ in $\mathcal{S}^2(\mathcal{H})$ and $A_n+K_n\to A+K$ resolvent strongly.

Assume that $f \in C^2(\mathbb{R})$ and that f'' is bounded. Then, Theorem 5.1 gives a meaning to $f(A+K) - f(A) - \frac{d}{dt}\Big(f(A+tK)\Big)\Big|_{t=0}$ as $\left[\Gamma^{A+K,A,A}(f^{[2]})\right](K,K)$. Moreover, the latter implies, by Corollary 5.8, that

$$f(A_n+K_n) - f(A_n) - \frac{d}{dt}\Big(f(A_n+tK_n)\Big)\Big|_{t=0} \underset{n\to+\infty}{\longrightarrow} \left[\Gamma^{A+K,A,A}(f^{[2]})\right](K,K)$$

in $\mathcal{S}^2(\mathcal{H})$. By the same corollary, we also know that this limit does not depend on the approximation of A by bounded operators $(A_n)_n$.
Moreover, we have

$$\begin{aligned} & f(A_n+K_n) - f(A_n) - \frac{d}{dt}\Big(f(A_n+tK_n)\Big)\Big|_{t=0} \\ & \quad = \left(\bigoplus_{i=1}^{n} f(\tilde{A}_i+\tilde{K}_i) - f(\tilde{A}_i) - \frac{d}{dt}\Big(f(\tilde{A}_i+t\tilde{K}_i)\Big)\Big|_{t=0}\right) \oplus \left(\bigoplus_{i=n+1}^{+\infty} 0_{\mathcal{H}_i}\right) \end{aligned}$$

and this sequence converges in $\mathcal{S}^2(\mathcal{H})$ to

$$\bigoplus_{i=1}^{+\infty}\left(f(\tilde{A}_i+\tilde{K}_i) - f(\tilde{A}_i) - \frac{d}{dt}\Big(f(\tilde{A}_i+t\tilde{K}_i)\Big)\Big|_{t=0}\right). \tag{5.26}$$

For both counterexamples to Peller's problems, the operators A and K will have this form. Note that according to (5.26), we have

$$\left\| f(A+K) - f(A) - \frac{d}{dt}\Big(f(A+tK)\Big)\Big|_{t=0} \right\|_1 \tag{5.27}$$

$$= \sum_{i=1}^{+\infty} \left\| \left(f(\tilde{A}_i+\tilde{K}_i) - f(\tilde{A}_i) - \frac{d}{dt}\Big(f(\tilde{A}_i+t\tilde{K}_i)\Big)\Big|_{t=0}\right) \right\|_1. \tag{5.28}$$

Therefore, the construction of a counterexample reduces to the construction of selfadjoint operators $\tilde{A}_i$ and $\tilde{K}_i$ acting on a finite dimensional Hilbert space such that

$$\left\| \left(f(\tilde{A}_i+\tilde{K}_i) - f(\tilde{A}_i) - \frac{d}{dt}\Big(f(\tilde{A}_i+t\tilde{K}_i)\Big)\Big|_{t=0}\right) \right\|_1$$

can be estimated from below in order to have a divergent series. To do so, we will use the connection between those operators and triple operator integrals (see Theorem 5.1). Using together (4.19) and Theorem 3.4, we can see that we have to estimate from below the $\mathcal{S}^1$-norm of Schur multipliers, for which some results (counterexamples) are known.

5.3 The self-adjoint case

5.3.1 A few properties of triple operator integrals

In this subsection, $\phi \colon \mathbb{R}^2 \to \mathbb{C}$ and $\psi \colon \mathbb{R}^3 \to \mathbb{C}$ denote arbitrary functions, and $n \in \mathbb{N}$ is a fixed integer. The following lemmas give some nice properties of triple operator integrals that we will use for our construction of a counter-example to Peller's problem.

Lemma 5.9. *Let $A \in \mathcal{B}(\mathbb{C}^n)$ be a self-adjoint operator and $X, Y \in \mathcal{B}(\mathbb{C}^n)$. Let*

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad \tilde{X} = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}.$$

Then

$$\Gamma^{\tilde{A},\tilde{A},\tilde{A}}(\psi)(\tilde{X},\tilde{X}) = \begin{pmatrix} \Gamma^{A,A,A}(\psi)(X,Y) & 0 \\ 0 & \Gamma^{A,A,A}(\psi)(Y,X) \end{pmatrix}.$$

Proof. Let $\{\lambda_i\}_{i=1}^m$ be the set of distinct eigenvalues of the operator A, $m \leq n$, and let E_i^A be the spectral projection of A associated with λ_i, $1 \leq i \leq m$. Clearly, the operator $\tilde{A}$ has the same set $\{\lambda_i\}_{i=1}^m$ of distinct eigenvalues and the spectral projection of the operator $\tilde{A}$ associated with λ_i is given by

$$E_i^{\tilde{A}} = \begin{pmatrix} E_i^A & 0 \\ 0 & E_i^A \end{pmatrix}, \quad 1 \leq i \leq m.$$

Therefore, we have

$$\begin{aligned} \Gamma^{\tilde{A},\tilde{A},\tilde{A}}(\psi)(\tilde{X},\tilde{X}) &= \sum_{i,k,j=1}^m \psi(\lambda_i,\lambda_k,\lambda_j) \begin{pmatrix} E_i^A & 0 \\ 0 & E_i^A \end{pmatrix} \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \times \\ &\qquad \begin{pmatrix} E_k^A & 0 \\ 0 & E_k^A \end{pmatrix} \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \begin{pmatrix} E_j^A & 0 \\ 0 & E_j^A \end{pmatrix} \\ &= \sum_{i,k,j=1}^m \psi(\lambda_i,\lambda_k,\lambda_j) \begin{pmatrix} E_i^A X E_k^A Y E_j^A & 0 \\ 0 & E_i^A Y E_k^A X E_j^A \end{pmatrix} \\ &= \begin{pmatrix} \Gamma^{A,A,A}(\psi)(X,Y) & 0 \\ 0 & \Gamma^{A,A,A}(\psi)(Y,X) \end{pmatrix}. \end{aligned}$$

□

Lemma 5.10. *Let $A, B \in \mathcal{B}(\mathbb{C}^n)$ be self-adjoint operators with the same set of eigenvalues and $X, Y \in \mathcal{B}(\mathbb{C}^n)$. Let*

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{Y} = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}.$$

Then

$$\Gamma^{\tilde{A},\tilde{A},\tilde{A}}(\psi)(\tilde{X},\tilde{Y}) = \begin{pmatrix} 0 & \Gamma^{A,B,B}(\psi)(X,Y) \\ 0 & 0 \end{pmatrix}.$$

Proof. Let $\{\lambda_i\}_{i=1}^m$ be the set of distinct eigenvalues of the operator A, $m \leq n$, and let E_i^A (resp. E_i^B) be the spectral projection of A (resp. B) associated with λ_i, $1 \leq i \leq m$. Since A and B have the same set of eigenvalues, the operator $\tilde{A}$ has the same set $\{\lambda_i\}_{i=1}^m$ of distinct eigenvalues and the spectral projection of the operator $\tilde{A}$ associated with λ_i is given by

$$E_i^{\tilde{A}} = \begin{pmatrix} E_i^A & 0 \\ 0 & E_i^B \end{pmatrix}, \quad 1 \leq i \leq m.$$

Therefore, we have

$$\begin{aligned}
\Gamma^{\tilde{A},\tilde{A},\tilde{A}}(\psi)(\tilde{X},\tilde{Y}) &= \sum_{i,k,j=1}^{m} \psi(\lambda_i,\lambda_k,\lambda_j) \begin{pmatrix} E_i^A & 0 \\ 0 & E_i^B \end{pmatrix} \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \times \\
&\qquad \begin{pmatrix} E_k^A & 0 \\ 0 & E_k^B \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} E_j^A & 0 \\ 0 & E_j^B \end{pmatrix} \\
&= \sum_{i,k,j=1}^{m} \psi(\lambda_i,\lambda_k,\lambda_j) \begin{pmatrix} 0 & E_i^A X E_k^B Y E_j^B \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Gamma^{A,B,B}(\psi)(X,Y) \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

□

Lemma 5.11. *Let $A_0, A_1, A_2 \in \mathcal{B}(\mathbb{C}^n)$ be self-adjoint operators. For any $a \neq 0 \in \mathbb{R}$ we have that*

$$\Gamma^{aA_0,aA_1,aA_2}(\psi) = \Gamma^{A_0,A_1,A_2}(\psi_a),$$

where

$$\psi_a(x_0,x_1,x_2) = \psi(ax_0,ax_1,ax_2), \quad x_0,x_1,x_2 \in \mathbb{R}.$$

Proof. Let $\{\lambda_i^{(j)}\}_{i=1}^{n_j}$ be the set of distinct eigenvalues of A_j, $j = 0,1,2$. Fix $a \neq 0$ in $\mathbb{R}$. It is clear that for any j, $\{a\lambda_i^{(j)}\}_{i=1}^{n_j}$ is the set of distinct eigenvalues of aA_j, and that the corresponding spectral projections coincide, that is, $E_i^{aA_j} = E_i^{A_j}$ for any $i = 1,\dots,n_j$. Therefore, for $X, Y \in \mathcal{B}(\mathbb{C}^n)$, we have

$$\begin{aligned}
\Gamma^{aA_0,aA_1,aA_2}(\psi)(X,Y) &= \sum_{i=1}^{n_0}\sum_{k=1}^{n_1}\sum_{j=1}^{n_2} \psi\big(a\lambda_i^{(0)}, a\lambda_k^{(1)}, a\lambda_j^{(2)}\big) E_i^{A_0} X E_k^{A_1} Y E_j^{A_2} \\
&= \Gamma^{A_0,A_1,A_2}(\psi_a)(X,Y).
\end{aligned}$$

□

Lemma 5.12. *Let $A, B \in \mathcal{B}(\mathbb{C}^n)$ be self-adjoint operators and let $\{U_m\}_{m\geq 1}$ be a sequence of unitary operators from $\mathcal{B}(\mathbb{C}^n)$ such that $U_m \to I_n$ as $m \to \infty$. Let also $X, Y \in \mathcal{B}(\mathbb{C}^n)$ and sequences $\{X_m\}_{m\geq 1}$ and $\{Y_m\}_{m\geq 1}$ in $\mathcal{B}(\mathbb{C}^n)$ such that $X_m \to X$ and $Y_m \to Y$ as $m \to \infty$. Let $\psi, \psi_m : \mathbb{R}^3 \to \mathbb{C}$ be functions such that $\psi_m \to \psi$ pointwise as $m \to \infty$. Then*

$$\Gamma^{U_mAU_m^*,B,B}(\psi_m)(X_m,Y_m) \longrightarrow \Gamma^{A,B,B}(\psi)(X,Y), \quad m \to \infty. \tag{5.29}$$

Proof. Let $\{\lambda_i\}_{i=1}^{m_0}$ and $\{\mu_k\}_{k=1}^{m_1}$ be the set of distinct eigenvalues of the operators A and B, respectively, $m_0, m_1 \leq n$, and let E_i^A (resp. E_k^B) be the spectral projection of A (resp. B) associated with λ_i (resp. μ_k), $1 \leq i \leq m_0$ (resp. $1 \leq k \leq m_1$). It is clear that the sequence $\{\lambda_i\}_{i=1}^{m_0}$ is the sequence of eigenvalues of $U_mAU_m^*$ and that the spectral projection of $U_mAU_m^*$ associated with λ_i is given by

$$E_i^{U_mAU_m^*} = U_m E_i^A U_m^*, \quad 1 \leq i \leq m_0.$$

Observe that

$$\begin{aligned}\Gamma^{U_m A U_m^*, B, B}(\psi_m)(X_m, Y_m) &= \sum_{i=1}^{m_0} \sum_{j,k=1}^{m_1} \psi_m(\lambda_i, \mu_k, \mu_j) E_i^{U_m A U_m^*} X E_k^B Y E_j^B \\ &= U_m \Big(\sum_{i=1}^{m_0} \sum_{j,k=1}^{m_1} \psi_m(\lambda_i, \mu_k, \mu_j) E_i^A (U_m^* X) E_k^B Y E_j^B \Big) \\ &= U_m \Gamma^{A,B,B}(\psi_m)(U_m^* X, Y).\end{aligned}$$

We claim that $\Gamma^{A,B,B}(\psi_m)(U_m^* X, Y) \to \Gamma^{A,B,B}(\psi)(X, Y)$. Indeed, we have

$$\begin{aligned}&\|\Gamma^{A,B,B}(\psi_m)(U_m^* X, Y) - \Gamma^{A,B,B}(\psi)(X, Y)\|_\infty \\ &\quad \leq \|\Gamma^{A,B,B}(\psi_m)(U_m^* X, Y) - \Gamma^{A,B,B}(\psi_m)(X, Y)\|_\infty \\ &\qquad + \|\Gamma^{A,B,B}(\psi_m)(X, Y) - \Gamma^{A,B,B}(\psi)(X, Y)\|_\infty \\ &\quad \leq \|\Gamma^{A,B,B}(\psi_m)(U_m^* X - X, Y)\|_\infty + \|\Gamma^{A,B,B}(\psi_m - \psi)(X, Y)\|_\infty \\ &\quad \leq \sum_{i=1}^{m_0} \sum_{j,k=1}^{m_1} |\psi_m(\lambda_i, \mu_k, \mu_j)| \|U_m X - X\|_\infty \|Y\|_\infty + \\ &\qquad\qquad \sum_{i=1}^{m_0} \sum_{j,k=1}^{m_1} |\psi_m - \psi|(\lambda_i, \mu_k, \mu_j) \|X\|_\infty \|Y\|_\infty.\end{aligned}$$

This upper bound tends to 0 as $m \to \infty$, which proves the claim.

Now since $U_m \to I_n$, we have

$$U_m \Gamma^{A,B,B}(\psi_m)(U_m^* X, Y) - \Gamma^{A,B,B}(\psi_m)(U_m^* X, Y) \longrightarrow 0$$

as $m \to \infty$. The result follows at once. □

Lemma 5.13. *Let $A \in \mathcal{B}(\mathbb{C}^n)$ be a self-adjoint operator and let $X \in \mathcal{B}(\mathbb{C}^n)$ commute with A.*

(i) We have

$$\Gamma^{A,A,A}(\psi)(X, X) = \widehat{\psi}(A) \times X^2, \quad X \in \mathcal{B}(\mathbb{C}^n),$$

where $\widehat{\psi} \colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\widehat{\psi}(x) = \psi(x, x, x), \quad x \in \mathbb{R}.$$

(ii) We have

$$\Gamma^{A,A,A}(\psi)(Y, X) = \Gamma^{A,A}(\phi_1)(Y) \times X, \quad Y \in \mathcal{B}(\mathbb{C}^n),$$

where

$$\phi_1(x_0, x_1) = \psi(x_0, x_1, x_1), \quad x_0, x_1 \in \mathbb{R}.$$

(iii) We have

$$\Gamma^{A,A,A}(\psi(X, Y)) = X \times \Gamma^{A,A}(\phi_2)(Y), \quad Y \in \mathcal{B}(\mathbb{C}^n),$$

where

$$\phi_2(x_0, x_1) = \psi(x_0, x_0, x_1), \quad x_0, x_1 \in \mathbb{R}.$$

Proof. Let $\{\xi_i\}_{i=1}^n$ be an orthonormal basis of eigenvectors of A and let $\{\lambda_i\}_{i=1}^n$ be the associated n-tuple of eigenvalues. Since A commutes with X, it follows that the projection P_{ξ_i} commutes with X for all $1 \le i \le n$. Thus, we have that

$$\begin{aligned}\Gamma^{A,A,A}(\psi)(X,X) &= \sum_{i,j,k=1}^{n} \psi(\lambda_i,\lambda_k,\lambda_j) P_{\xi_i} X P_{\xi_k} X P_{\xi_j} \\ &= \sum_{i=1}^{n} \psi(\lambda_i,\lambda_i,\lambda_i) P_{\xi_i} \times X^2 \\ &= \sum_{i=1}^{n} \widehat{\psi}(\lambda_i) P_{\xi_i} \times X^2 = \widehat{\psi}(A) \times X^2,\end{aligned}$$

which proves (i).

Similarly, for (ii), we have

$$\begin{aligned}\Gamma^{A,A,A}(\psi)(Y,X) &= \sum_{i,j,k=1}^{n} \psi(\lambda_i,\lambda_k,\lambda_j) P_{\xi_i} Y P_{\xi_k} X P_{\xi_j} \\ &= \sum_{i,k=1}^{n} \psi(\lambda_i,\lambda_k,\lambda_k) P_{\xi_i} Y P_{\xi_k} \times X \\ &= \sum_{i,k=1}^{n} \phi_1(\lambda_i,\lambda_k) P_{\xi_i} Y P_{\xi_k} \times X = \Gamma^{A,A}(\phi_1)(Y) \times X.\end{aligned}$$

The proof of (iii) repeats that of (ii). □

5.3.2 Finite-dimensional constructions

In this section we establish various estimates concerning finite dimensional operators. The symbol const will stand for uniform positive constants, not depending on the dimension.

It will be convenient to extend the definition of the divided difference of first order as follow: let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that f admits right and left derivatives $f'_r(x)$ and $f'_l(x)$ at each $x \in \mathbb{R}$. Assume further that f'_r, f'_l are bounded. The divided difference of the first order is defined by

$$f^{[1]}(x_0,x_1) := \begin{cases} \frac{f(x_0)-f(x_1)}{x_0-x_1}, & \text{if } x_0 \neq x_1 \\ \frac{f'_r(x_0)+f'_l(x_0)}{2} & \text{if } x_0 = x_1 \end{cases}, \quad x_0, x_1 \in \mathbb{R}.$$

Then $f^{[1]}$ is a bounded Borel function.

If f is C^2-function, the definition of the second divided difference $f^{[2]}$ is given in Section 5.2. $f^{[2]}$ is a bounded continuous function, with

$$\left\| f^{[2]} \right\| = \frac{1}{2} \left\| f'' \right\|_\infty. \tag{5.30}$$

Consider the function $f_0\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f_0(x) = |x|, \qquad x \in \mathbb{R}.$$

The definition of $f_0^{[1]}$ given above applies to this function.

The following result is proved in [Dav88, Theorem 13].

Theorem 5.14. *For all $n \in \mathbb{N}$ there exist self-adjoint operators $A_n, B_n \in \mathcal{B}(\mathbb{C}^{2n+1})$ such that the spectra of $A_n + B_n$ and A_n coincide, 0 is an eigenvalue of A_n, and*

$$\|f_0(A_n + B_n) - f_0(A_n)\|_1 \geq \text{const } \log n \|B_n\|_1. \tag{5.31}$$

Remark 5.15. *The operator A_n constructed in [Dav88] is a diagonal operator defined on $\mathbb{C}^{2n}$ and 0 is not an eigenvalue of A_n. By changing the dimension from $2n$ to $2n+1$ and adding a zero on the diagonal, one obtains the operator A_n in Theorem 5.14, with 0 in the spectrum.*

Corollary 5.16. *For all $n \geq 1$, there exist self-adjoint operators $A_n, B_n \in \mathcal{B}(\mathbb{C}^{2n+1})$ such that the spectra of $A_n + B_n$ and A_n coincide, and*

$$\left\|\Gamma^{A_n+B_n, A_n}(f_0^{[1]}) : \mathcal{S}^\infty_{2n+1} \to \mathcal{S}^\infty_{2n+1}\right\| \geq \text{const } \log n.$$

Proof. Take $A_n, B_n \in \mathcal{B}(\mathbb{C}^{2n+1})$ as in Theorem 5.14. By (5.9), we have that

$$\Gamma^{A_n+B_n, A_n}(f_0^{[1]})(B_n) = f_0(A_n + B_n) - f_0(A_n).$$

By Theorem 5.14, we have that

$$\|\Gamma^{A_n+B_n, A_n}(f_0^{[1]})(B_n)\|_1 = \|f_0(A_n + B_n) - f_0(A_n)\|_1 \geq \text{const } \log n \|B_n\|_1.$$

Therefore,

$$\left\|\Gamma^{A_n+B_n, A_n}(f_0^{[1]}) : \mathcal{S}^1_{2n+1} \to \mathcal{S}^1_{2n+1}\right\| \geq \text{const } \log n.$$

Since the operator $\Gamma^{A_n+B_n, A_n}(f_0^{[1]})$ is a Schur multiplier, we obtain that

$$\left\|\Gamma^{A_n+B_n, A_n}(f_0^{[1]}) : \mathcal{S}^\infty_{2n+1} \to \mathcal{S}^\infty_{2n+1}\right\| \geq \text{const } \log n.$$

□

Consider the function $g_0 : \mathbb{R} \to \mathbb{R}$ given by

$$g_0(x) = x|x| = x f_0(x), \qquad x \in \mathbb{R}.$$

This is a C^1-function. Hence although g_0 is not a C^2-function, one may define $g_0^{[2]}(x_0, x_1, x_2)$ by the formula given in (5.2) and in the beginning of this subsection whenever x_0, x_1, x_2 are not equal. Let us define

$$\psi_0(x_0, x_1, x_2) := \begin{cases} g_0^{[2]}(x_0, x_1, x_2), & \text{if } x_0 \neq x_1 \text{ or } x_1 \neq x_2 \\ 1, & \text{if } x_0 = x_1 = x_2 > 0 \\ -1, & \text{if } x_0 = x_1 = x_2 < 0 \\ 0, & \text{if } x_0 = x_1 = x_2 = 0 \end{cases}.$$

The following lemma relates the linear Schur multiplier for $f_0^{[1]}$ and the bilinear Schur multiplier for ψ_0.

Lemma 5.17. *For self-adjoint operators $A_n, B_n \in \mathcal{B}(\mathbb{C}^n)$ such that 0 belongs to the spectrum of A_n, the inequality*

$$\left\|\Gamma^{A_n+B_n,A_n,A_n}(\psi_0) : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\right\| \geq \left\|\Gamma^{A_n+B_n,A_n}(f_0^{[1]}) : \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty\right\| \tag{5.32}$$

holds.

Proof. Let $\{\mu_k\}_{k=1}^n$ be the sequence of eigenvalues of the operator A_n. For simplicity, we assume that $\mu_1 = 0$.

By formulas (4.16) and (4.19) and by Theorem 3.4, we have that

$$\left\|\Gamma^{A_n+B_n,A_n,A_n}(\psi_0) : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\right\| = \max_{1\leq k\leq n} \|\Gamma^{A_n+B_n,A_n}(\varphi_k) : \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty\|,$$

where

$$\varphi_k(x_0,x_1) := \psi_0(x_0,\mu_k,x_1), \quad x_0,x_1 \in \mathbb{R},\ 1 \leq k \leq n.$$

This implies

$$\left\|\Gamma^{A_n+B_n,A_n,A_n}(\psi_0) : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\right\| \geq \|\Gamma^{A_n+B_n,A_n}(\varphi_1) : \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty\|.$$

It therefore suffices to check that

$$\varphi_1 = f_0^{[1]}. \tag{5.33}$$

By definition, $\varphi_1 = \psi_0(\cdot\,,0,\cdot\,)$. In particular,

$$\varphi_1(0,0) = \psi_0(0,0,0) = 0 = f_0^{[1]}(0,0).$$

Consider now $(x_0,x_1) \in \mathbb{R}^2$ such that $x_0 \neq 0$ or $x_1 \neq 0$. In that case, we have

$$\varphi_1(x_0,x_1) = g_0^{[2]}(x_0,0,x_1).$$

If $x_0, x_1, 0$ are mutually distinct, then

$$\begin{aligned} g_0^{[2]}(x_0,0,x_1) &= \frac{g_0^{[1]}(x_0,0) - g_0^{[1]}(0,x_1)}{x_0 - x_1} = \frac{\frac{x_0 f_0(x_0)-0}{x_0-0} - \frac{0-x_1 f_0(x_1)}{0-x_1}}{x_0 - x_1} \\ &= \frac{f_0(x_0) - f_0(x_1)}{x_0 - x_1} = f_0^{[1]}(x_0,x_1). \end{aligned}$$

If $x_0 = 0$ and $x_1 \neq 0$, then

$$\begin{aligned} g_0^{[2]}(0,0,x_1) &= \frac{g_0^{[1]}(0,0) - g_0^{[1]}(0,x_1)}{x_0 - x_1} = \frac{g_0'(0) - \frac{0-x_1 f_0(x_1)}{0-x_1}}{0 - x_1} \\ &= \frac{f_0(x_1)}{x_1} = f_0^{[1]}(0,x_1). \end{aligned}$$

The argument is similar, when $x_0 \neq 0$ and $x_1 = 0$.

Assume now that $x_0 = x_1 \neq 0$. Then we have

$$\begin{aligned} g_0^{[2]}(x_0, 0, x_0) &= \frac{d}{dx} g_0^{[1]}(x, 0)\Big|_{x=x_0} = \frac{d}{dx}\Big(\frac{x f_0(x) - 0}{x - 0}\Big)\Big|_{x=x_0} \\ &= f_0'(x_0) = f_0^{[1]}(x_0, x_0). \end{aligned}$$

This completes the proof of (5.33) and we obtain (5.32). □

The following is a straightforward consequence of Corollary 5.16 and Lemma 5.17.

Corollary 5.18. *For every $n \geq 1$ there exist self-adjoint operators $A_n, B_n \in \mathcal{B}(\mathbb{C}^{2n+1})$ such that the spectra of $A_n + B_n$ and A_n coincide, and*

$$\left\| \Gamma^{A_n+B_n, A_n, A_n}(\psi_0) : \mathcal{S}^2_{2n+1} \times \mathcal{S}^2_{2n+1} \to \mathcal{S}^1_{2n+1} \right\| \geq \text{const } \log n.$$

We assume below that $n \geq 1$ is fixed and that A_n, B_n are given by Corollary 5.18. The purpose of the series of Lemmas 5.19-5.24 below is to prove Lemma 5.25, which is the final step in the finite-dimensional resolution of Peller's problem. The following result follows immediately from Corollary 5.18.

Lemma 5.19. *There are operators $X_n, Y_n \in \mathcal{B}(\mathbb{C}^{2n+1})$ with $\|X_n\|_2 = \|Y_n\|_2 = 1$, such that*

$$\left\| \Gamma^{A_n+B_n, A_n, A_n}(\psi_0)(X_n, Y_n) \right\|_1 \geq \text{const } \log n.$$

Let us denote

$$H_n := \begin{pmatrix} A_n + B_n & 0 \\ 0 & A_n \end{pmatrix} \tag{5.34}$$

and consider the operator

$$T_1 := \Gamma^{H_n, H_n, H_n}(\psi_0) \colon \mathcal{S}^2_{4n+2} \times \mathcal{S}^2_{4n+2} \to \mathcal{S}^1_{4n+2}.$$

Lemma 5.20. *There are operators $\tilde{X}_n, \tilde{Y}_n \in \mathcal{B}(\mathbb{C}^{4n+2})$ with $\|\tilde{X}_n\|_2 = \|\tilde{Y}_n\|_2 = 1$, such that*

$$\left\| T_1(\tilde{X}_n, \tilde{Y}_n) \right\|_1 \geq \text{const } \log n.$$

Proof. Take

$$\tilde{X}_n := \begin{pmatrix} 0 & X_n \\ 0_{2n+1} & 0 \end{pmatrix}, \quad \tilde{Y}_n := \begin{pmatrix} 0_{2n+1} & 0 \\ 0 & Y_n \end{pmatrix},$$

where X_n, Y_n are operators from Lemma 5.19 and 0_{2n+1} is the null element of $\mathcal{B}(\mathbb{C}^{2n+1})$. Clearly, $\|\tilde{X}_n\|_2 = \|X_n\|_2 = 1$ and $\|\tilde{Y}_n\|_2 = \|Y_n\|_2 = 1$. It follows from Lemma 5.10 and the fact that $A_n + B_n$ and A_n have the same spectra that

$$T_1(\tilde{X}_n, \tilde{Y}_n) = \begin{pmatrix} 0 & \Gamma^{A_n+B_n, A_n, A_n}(\psi_0)(X_n, Y_n) \\ 0_{2n+1} & 0 \end{pmatrix}.$$

Therefore, by Lemma 5.19,

$$\left\| T_1(\tilde{X}_n, \tilde{Y}_n) \right\|_1 = \left\| \Gamma^{A_n+B_n, A_n, A_n}(\psi_0)(X_n, Y_n) \right\|_1 \geq \text{const } \log n.$$

□

Lemma 5.21. *There is an operator $S_n \in \mathcal{B}(\mathbb{C}^{4n+2})$ with $\|S_n\|_2 \leq 1$ such that*

$$\left\|T_1(S_n, S_n^*)\right\|_1 \geq \text{const}\ \log n.$$

Proof. Take the operators $\tilde{X}_n, \tilde{Y}_n \in \mathcal{B}(\mathbb{C}^{4n+2})$ as in Lemma 5.20. By the polarization identity

$$T_1(\tilde{X}_n, \tilde{Y}_n) = \frac{1}{4}\sum_{k=0}^{3} i^k T_1((\tilde{X}_n + i^k\tilde{Y}_n^*), (\tilde{X}_n + i^k\tilde{Y}_n^*)^*),$$

we have that

$$\|T_1(\tilde{X}_n, \tilde{Y}_n)\|_1 \leq \max_{0\leq k\leq 3} \|T_1((\tilde{X}_n + i^k\tilde{Y}_n^*), (\tilde{X}_n + i^k\tilde{Y}_n^*)^*)\|_1.$$

Taking k_0 such that

$$\|T_1((\tilde{X}_n + i^{k_0}\tilde{Y}_n^*), (\tilde{X}_n + i^{k_0}\tilde{Y}_n^*)^*)\|_1 = \max_{0\leq k\leq 3} \|T_1((\tilde{X}_n + i^k\tilde{Y}_n^*), (\tilde{X}_n + i^k\tilde{Y}_n^*)^*)\|_1,$$

we set

$$S_n := \frac{1}{2}(\tilde{X}_n + i^{k_0}\tilde{Y}_n^*).$$

Thus, by Lemma 5.20, we have

$$\left\|T_1(S_n, S_n^*)\right\|_1 \geq \frac{1}{4}\|T_1(\tilde{X}_n, \tilde{Y}_n)\|_1 \geq \text{const}\ \log n$$

and

$$\|S_n\|_2 \leq \frac{1}{2}(\|\tilde{X}_n\|_2 + \|\tilde{Y}_n\|_2) = 1.$$

□

Let us denote

$$\tilde{H}_n := \begin{pmatrix} H_n & 0 \\ 0 & H_n \end{pmatrix} = \begin{pmatrix} A_n + B_n & 0 & 0 & 0 \\ 0 & A_n & 0 & 0 \\ 0 & 0 & A_n + B_n & 0 \\ 0 & 0 & 0 & A_n \end{pmatrix}, \quad n \geq 1, \tag{5.35}$$

and consider the operator

$$T_2 := \Gamma^{\tilde{H}_n, \tilde{H}_n, \tilde{H}_n}(\psi_0) \colon \mathcal{S}^2_{8n+4} \times \mathcal{S}^2_{8n+4} \to \mathcal{S}^1_{8n+4}.$$

Lemma 5.22. *There is a self-adjoint operator $Z_n \in \mathcal{B}(\mathbb{C}^{8n+4})$ with $\|Z_n\|_2 \leq 1$ such that*

$$\left\|T_2(Z_n, Z_n)\right\|_1 \geq \text{const}\ \log n.$$

Proof. Consider the operator S_n from Lemma 5.21. Setting

$$Z_n := \frac{1}{2}\begin{pmatrix} 0 & S_n \\ S_n^* & 0 \end{pmatrix},$$

we have $\|Z_n\|_2 = \frac{1}{2}(\|S_n\|_2 + \|S_n^*\|_2) \le 1$ and by Lemma 5.9,

$$T_2(Z_n, Z_n) = \frac{1}{4}\begin{pmatrix} T_1(S_n, S_n^*) & 0 \\ 0 & T_1(S_n^*, S_n) \end{pmatrix}.$$

Therefore, by Lemma 5.21, we arrive at

$$\begin{aligned}\left\|T_2(Z_n, Z_n)\right\|_1 &= \frac{1}{4}\left(\left\|T_1(S_n, S_n^*)\right\|_1 + \left\|T_1(S_n^*, S_n)\right\|_1\right) \\ &\ge \frac{1}{4}\left\|T_1(S_n, S_n^*)\right\|_1 \ge \text{const } \log n.\end{aligned}$$

□

The following decomposition principle is of independent interest. In this statement we use the notation $[H, F] = HF - FH$ for the commutator of H and F.

Lemma 5.23. *For any self-adjoint operators $Z, H \in \mathcal{B}(\mathbb{C}^n)$, there are self-adjoint operators $F, G \in \mathcal{B}(\mathbb{C}^n)$ such that*

$$Z = G + i[H, F],$$

the matrix G commutes with H, and we have

$$\|[H, F]\|_2 \le 2\ \|Z\|_2 \ \ \textit{and} \ \ \|G\|_2 \le \|Z\|_2\,.$$

Proof. Let

$$h_1, h_2, \dots, h_m$$

be the pairwise distinct eigenvalues of the operator H and let

$$E_1, E_2, \dots, E_m$$

be the associated spectral projections, so that

$$H = \sum_{j=1}^{m} h_j E_j.$$

We set

$$G = \sum_{j=1}^{m} E_j Z E_j \qquad \text{and} \qquad F = i \sum_{\substack{j=1 \\ j \neq k}}^{m} (h_k - h_j)^{-1}\, E_j Z E_k.$$

Since

$$HE_j = h_j E_j,$$

we have

$$[H, E_j Z E_k] = \ H \times E_j Z E_k - \ E_j Z E_k \times H = (h_j - h_k) \times E_j Z E_k.$$

Consequently,

$$i[H, F] = \sum_{\substack{j=1 \\ j \neq k}}^{m} E_j Z E_k$$

and hence

$$G + i[H, F] = Z.$$

Further F, G are self-adjoint and it is clear that $[G, H] = 0$. Hence the first two claims of the lemma are proved.

Now take

$$U_t = \sum_{j=1}^{m} e^{ijt} E_j, \quad t \in [-\pi, \pi].$$

Then

$$\int_{-\pi}^{\pi} U_t Z U_t^* \frac{\mathrm{d}t}{2\pi} = \sum_{j,k=1}^{m} E_j Z E_k \int_{-\pi}^{\pi} e^{i(j-k)\,t} \frac{\mathrm{d}t}{2\pi} = \sum_{j=1}^{m} E_j Z E_j = G.$$

Since U_t is unitary, we deduce that

$$\|G\|_2 \le \int_{-\pi}^{\pi} \|U_t Z U_t^*\|_2 \frac{\mathrm{d}t}{2\pi} \le \|Z\|_2 .$$

Moreover writing

$$i[H, F] = Z - G$$

we deduce that

$$\|[H, F]\|_2 \le 2 \|Z\|_2 .$$

□

Lemma 5.24. *There is a self-adjoint operator $F_n \in \mathcal{B}(\mathbb{C}^{8n+4})$ such that $\|[\tilde{H}_n, F_n]\|_2 \le 2$ and*

$$\left\| T_2\big(i[\tilde{H}_n, F_n], i[\tilde{H}_n, F_n]\big) \right\|_1 \ge \text{const } \log n - 5.$$

Proof. Take the operator Z_n in $\mathcal{B}(\mathbb{C}^{8n+4})$ given by Lemma 5.22. By Lemma 5.23, we may choose self-adjoint operators F_n and G_n from $\mathcal{B}(\mathbb{C}^{8n+4})$ such that

$$Z_n = G_n + i[\tilde{H}_n, F_n], \quad [G_n, \tilde{H}_n] = 0,$$

and

$$\|[\tilde{H}_n, F_n]\|_2 \le 2 \; \|Z_n\|_2 , \qquad \|G_n\|_2 \le \|Z_n\|_2 . \tag{5.36}$$

We compute

$$\begin{aligned} T_2(Z_n, Z_n) &= T_2\Big(G_n + i[\tilde{H}_n, F_n],\, G_n + i[\tilde{H}_n, F_n]\Big) \\ &= T_2\Big(G_n,\, G_n\Big) \\ &+ T_2\Big(G_n,\, i[\tilde{H}_n, F_n]\Big) \\ &+ T_2\Big(i[\tilde{H}_n, F_n],\, G_n\Big) \\ &+ T_2\Big(i[\tilde{H}_n, F_n],\, i[\tilde{H}_n, F_n]\Big). \end{aligned} \tag{5.37}$$

We shall estimate the first three summands above. We apply Lemma 5.13 to the function ψ_0 and use the notation from the latter statement. The operator G_n commutes with $\tilde{H}_n$ hence by the first part of Lemma 5.13,

$$T_2(G_n, G_n) = \widehat{\psi_0}(\tilde{H}_n) \times G_n^2.$$

Furthermore $\widehat{\psi_0}(x) = 1$ if $x > 0$, $\widehat{\psi_0}(x) = -1$ if $x < 0$ and $\widehat{\psi_0}(0) = 0$. Hence

$$\|\widehat{\psi_0}(\tilde{H}_n)\|_\infty \leq 1.$$

This implies that

$$\big\|T_2(G_n, G_n)\big\|_1 \leq \|\widehat{\psi_0}(\tilde{H}_n)\|_\infty \|G_n\|_2^2 \leq \|Z_n\|_2^2 \leq 1.$$

Next applying the second and third part of Lemma 5.13, we obtain

$$T_2\Big(i[\tilde{H}_n, F_n],\, G_n\Big) = i\Gamma^{\tilde{H}_n, \tilde{H}_n}(\phi_1)\Big([\tilde{H}_n, F_n]\Big) \times G_n$$

and

$$T_2\Big(G_n, i[\tilde{H}_n, F_n]\Big) = i\; G_n \times \Gamma^{\tilde{H}_n, \tilde{H}_n}(\phi_2)\Big([\tilde{H}_n, F_n]\Big),$$

where

$$\phi_1(x_0, x_1) \;=\; \psi_0(x_0, x_1, x_1) \quad \text{and} \quad \phi_2(x_0, x_1) \;=\; \psi_0(x_0, x_0, x_1), \qquad x_0, x_1 \;\in\; \mathbb{R}.$$

We have $g_0' = 2|\cdot|$ hence if $x_0 \neq x_1$, we have

$$\begin{aligned}(x_0 - x_1)\phi_1(x_0, x_1) &= \frac{g_0(x_0) - g_0(x_1)}{x_0 - x_1} - g_0'(x_1) \\ &= 2\Big(\int_0^1 \big|tx_0 + (1-t)x_1\big|\, dt \,-\, |x_1|\Big).\end{aligned}$$

Using the elementary inequality $\big||z| - |z'|\big| \leq |z - z'|$, we deduce that $|\phi_1(x_0, x_1)| \leq 1$. This implies that $\|\phi_1\|_\infty \leq 1$. Consequently

$$\begin{aligned}\Big\|\Gamma^{\tilde{H}_n, \tilde{H}_n}(\phi_1)\Big([\tilde{H}_n, F_n]\Big) \times G_n\Big\|_1 &\leq \Big\|\Gamma^{\tilde{H}_n, \tilde{H}_n}(\phi_1)\Big([\tilde{H}_n, F_n]\Big)\Big\|_2 \|G_n\|_2 \\ &\leq \|\phi_1\|_\infty \|[\tilde{H}_n, F_n]\|_2 \|G_n\|_2 \\ &\leq 2\|\phi_1\|_\infty \|Z_n\|_2^2 \leq 2\end{aligned}$$

by (5.36) and Lemma 5.22. Similarly, $\|\phi_2\|_\infty \leq 1$ and

$$\Big\|G_n \times \Gamma^{\tilde{H}_n, \tilde{H}_n}(\phi_2)\Big([\tilde{H}_n, F_n]\Big)\Big\|_1 \leq 2.$$

Combining the preceding estimates with (5.37), we arrive at

$$\|T_2(Z_n, Z_n)\|_1 \leq 5 + \Big\|T_2\Big(i[\tilde{H}_n, F_n],\, i[\tilde{H}_n, F_n]\Big)\Big\|_1.$$

Applying Lemma 5.22, we deduce the result. □

Lemma 5.25. *There exists a C^2-function g with a bounded second derivative and there exists $N \in \mathbb{N}$ such that for any sequence $\{\alpha_n\}_{n\geq N}$ of positive real numbers there is a sequence of operators $\tilde{B}_n \in \mathcal{B}(\mathbb{C}^{8n+4})$ such that $\|\tilde{B}_n\|_2 \leq 4\alpha_n$, for all $n \geq N$, and*

$$\|\Gamma^{\tilde{A}_n+\tilde{B}_n,\tilde{A}_n,\tilde{A}_n}(g^{[2]})(\tilde{B}_n,\tilde{B}_n)\|_1 \geq \text{const}\,\alpha_n^2\,\log n,\quad n \geq N.$$

Proof. Changing the constant 'const' in Lemma 5.24 by half of its value, we can change the estimate from that statement into

$$\left\|T_2\big(i[\tilde{H}_n,F_n],i[\tilde{H}_n,F_n]\big)\right\|_1 \geq \text{const}\,\log n,\quad n \geq N, \tag{5.38}$$

for sufficiently large $N \in \mathbb{N}$.

Take an arbitrary sequence $\{\alpha_n\}_{n\geq N}$ of positive real numbers, take the operator F_n from Lemma 5.24 and denote

$$\tilde{F}_n := \alpha_n F_n.$$

For any $t > 0$, consider

$$\gamma_t(\tilde{H}_n) = e^{it\tilde{F}_n}\tilde{H}_n e^{-it\tilde{F}_n},\quad \text{and}\quad V_{n,t} := \frac{\gamma_t(\tilde{H}_n)-\tilde{H}_n}{t}.$$

On one hand, it follows from the identity $\frac{d}{dt}(e^{it\tilde{F}_n})|_{t=0} = i\tilde{F}_n$ that

$$V_{n,t} \longrightarrow i[\tilde{F}_n,\tilde{H}_n],\quad t \to +0.$$

It therefore follows from Lemma 5.24 that there is $t_1 > 0$ such that

$$\|V_{n,t}\|_2 \leq 2\|[\tilde{F}_n,\tilde{H}_n]\|_2 = 2\alpha_n\|[F_n,\tilde{H}_n]\|_2 \leq 4\alpha_n \tag{5.39}$$

for all $t \leq t_1$. On the other hand,

$$\tilde{H}_n + t\,V_{n,t} = \gamma_t(\tilde{H}_n) \longrightarrow \tilde{H}_n,\quad t \to +0. \tag{5.40}$$

Take a C^2-function g such that $g(x) = g_0(x) = x|x|$ for $|x| > 1$ and $g^{(j)}(0) = 0$, $j = 0,1,2$. Denote

$$g_t(x_0,x_1,x_2) := g^{[2]}\Big(\frac{x_0}{t},\frac{x_1}{t},\frac{x_2}{t}\Big),\quad t > 0,\quad x_0,x_1,x_2 \in \mathbb{R}.$$

We claim that

$$\lim_{t\to+0} g_t(x_0,x_1,x_2) = \psi_0(x_0,x_1,x_2),\quad x_0,x_1,x_2 \in \mathbb{R}. \tag{5.41}$$

To prove this claim, we first observe, using the definition of g_0, that

$$\psi_0\Big(\frac{x_0}{t},\frac{x_1}{t},\frac{x_2}{t}\Big) = \psi_0(x_0,x_1,x_2),\quad x_0,x_1,x_2 \in \mathbb{R},\ t > 0. \tag{5.42}$$

Next we note that for any $x \in \mathbb{R}$,

$$g\Big(\frac{x}{t}\Big) = g_0\Big(\frac{x}{t}\Big)\quad \text{and}\quad g'\Big(\frac{x}{t}\Big) = g_0'\Big(\frac{x}{t}\Big)$$

for $t > 0$ small enough. For $x = 0$, this follows from the fact that by assumption, $g(0) = g'(0) = 0$. From these properties, we deduce that for any $x_0, x_1 \in \mathbb{R}$,

$$g^{[1]}\Big(\frac{x_0}{t}, \frac{x_1}{t}\Big) = g_0^{[1]}\Big(\frac{x_0}{t}, \frac{x_1}{t}\Big)$$

for $t > 0$ small enough.

In turn, this implies that if $x_0 \neq x_1$ or $x_1 \neq x_2$, then

$$g^{[2]}\Big(\frac{x_0}{t}, \frac{x_1}{t}, \frac{x_2}{t}\Big) = g_0^{[2]}\Big(\frac{x_0}{t}, \frac{x_1}{t}, \frac{x_2}{t}\Big)$$

for $t > 0$ small enough. According to (5.42), this implies that

$$g^{[2]}\Big(\frac{x_0}{t}, \frac{x_1}{t}, \frac{x_2}{t}\Big) = \psi_0(x_0, x_1, x_2)$$

for $t > 0$ small enough.

Consider now the case when $x_0 = x_1 = x_2$. For any $t > 0$, we have

$$g^{[2]}\Big(\frac{x_0}{t}, \frac{x_0}{t}, \frac{x_0}{t}\Big) = \frac{1}{2}\, g''\Big(\frac{x_0}{t}\Big).$$

If $x_0 > 0$, then $g''\big(\frac{x_0}{t}\big) = 2$ for $t > 0$ small enough, and if $x_0 < 0$, then $g''\big(\frac{x_0}{t}\big) = -2$ for $t > 0$ small enough. Furthermore, $g''(0) = 0$ by assumption. Hence

$$g^{[2]}\Big(\frac{x_0}{t}, \frac{x_0}{t}, \frac{x_0}{t}\Big) = \psi_0(x_0, x_0, x_0)$$

for $t > 0$ small enough. This completes the proof of (5.41).

Applying subsequently Lemma 5.11 with $a = \frac{1}{t}$, property (5.40) and Lemma 5.12, we obtain that

$$\begin{aligned}\Gamma^{\frac{1}{t}\tilde{H}_n + V_{n,t}, \frac{1}{t}\tilde{H}_n, \frac{1}{t}\tilde{H}_n}(g^{[2]})(V_{n,t}, V_{n,t}) &= \Gamma^{\tilde{H}_n + tV_{n,t}, \tilde{H}_n, \tilde{H}_n}(g_t)(V_{n,t}, V_{n,t}) \\ &\longrightarrow T_2\big(i[\tilde{F}_n, \tilde{H}_n], i[\tilde{F}_n, \tilde{H}_n]\big)\end{aligned}$$

when $t \to +0$. Furthermore,

$$T_2\big(i[\tilde{F}_n, \tilde{H}_n], i[\tilde{F}_n, \tilde{H}_n]\big) = \alpha_n^2 T_2\big(i[F_n, \tilde{H}_n], i[F_n, \tilde{H}_n]\big).$$

By (5.38), there is $t_2 > 0$ such that

$$\Big\|\Gamma^{\frac{1}{t}\tilde{H}_n + V_{n,t}, \frac{1}{t}\tilde{H}_n, \frac{1}{t}\tilde{H}_n}(g^{[2]})(V_{n,t}, V_{n,t})\Big\|_1 \geq \text{const}\ \alpha_n^2\ \log n$$

for all $t \leq t_2$. Taking $t_n = \min\{t_1, t_2\}$, and setting

$$\tilde{A}_n := \frac{1}{t_n}\tilde{H}_n, \quad \tilde{B}_n := V_{n,t_n},$$

we obtain that $\|\tilde{B}_n\|_2 \leq 4\alpha_n$ (see (5.39)) and

$$\|\Gamma^{\tilde{A}_n+\tilde{B}_n,\tilde{A}_n,\tilde{A}_n}(g^{[2]})(\tilde{B}_n,\tilde{B}_n)\|_1 \geq \text{const}\,\alpha_n^2\ \log n,$$

for all $n \geq N$. □

5.3.3 A solution to Peller's problem for selfadjoint operators

The following theorem answers Peller's problem (5.5) in negative.

Theorem 5.26. *There exists a function $f \in C^2(\mathbb{R})$ with a bounded second derivative, a self-adjoint operator A on $\mathcal{H}$ and a self-adjoint $B \in \mathcal{S}^2(\mathcal{H})$ as above such that*

$$f(A+B) - f(A) - \frac{d}{dt}\Big(f(A+tB)\Big)\Big|_{t=0} \notin \mathcal{S}^1(\mathcal{H}).$$

Proof. Take the integer $N \in \mathbb{N}$, the operators $\tilde{A}_n$, $\tilde{B}_n$ and the function g from Lemma 5.25, applied with the sequence $\{\alpha_n\}_{n\geq N}$ defined by

$$\alpha_n = \frac{1}{\sqrt{n\ \log^{3/2} n}}\,.$$

Let $\mathcal{H}_n = \ell^2_{8n+4}$ and let $\mathcal{H} = \overset{2}{\oplus}_{n\geq N}\mathcal{H}_n$. Then let $A = \oplus_{n=N}^{\infty} A_n$ and $B = \oplus_{n=N}^{\infty} B_n$ be the corresponding direct sums. Then the self-adjoint operator B belongs to $\mathcal{S}^2(\mathcal{H})$. Indeed, it follows from (5.25) and Lemma 5.25 that

$$\|B\|_2^2 = \sum_{n=N}^{\infty} \|\tilde{B}_n\|_2^2 \leq 16 \sum_{n=N}^{\infty} \alpha_n^2 = \sum_{n=N}^{\infty} \frac{16}{n\ \log^{3/2} n} < \infty.$$

On the other hand, by (5.27) and Lemma 5.25, we have

$$\begin{aligned}
\Big\|g(A+B)-g(A) - \frac{d}{dt}\big(g(A+tB)\big)\Big|_{t=0}\Big\|_1 \\
&= \sum_{n=N}^{\infty} \Big\|g(\tilde{A}_n+\tilde{B}_n) - g(\tilde{A}_n) - \frac{d}{dt}\big(g(\tilde{A}+t\tilde{B}_n)\big)\Big|_{t=0}\Big\|_1 \\
&= \sum_{n=N}^{\infty} \Big\|\Gamma^{\tilde{A}_n+\tilde{B}_n,\tilde{A}_n,\tilde{A}_n}(g^{[2]})\big(\tilde{B}_n,\,\tilde{B}_n\big)\Big\|_1 \\
&\geq \text{const} \sum_{n=N}^{\infty} \alpha_n^2\ \log n \\
&= \text{const} \sum_{n=N}^{\infty} \frac{1}{n\ \log^{1/2} n} = \infty.
\end{aligned}$$

□

Note that this theorem has been generalized in [DPT16]. The authors proved that for any $n \in \mathbb{N}$, there exist a function $f_n \in C^n(\mathbb{R})$, a separable Hilbert space $\mathcal{H}$ and selfadjoint operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{S}^n(\mathcal{H})$ such that

$$f_n(A+B) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k} \Big(f(A+tB) \Big) \Big|_{t=0} \notin \mathcal{S}^1(\mathcal{H}).$$

In this result, the operator A is bounded. For the case $n = 2$, the function f_2 is different from the one considered in this section. Their starting point was a C^1-function with a bad behavior on $\mathcal{B}(\ell_2)$. Therefore, they did not have to deal with the difficulty of the non-differentiability of the absolute value in 0. This is how they could obtain a bounded operator A.

5.4 The unitary case

5.4.1 Preliminary results

In this subsection we will consider, for a fixed integer n, $U_1, U_2, U_3 \in \mathcal{B}(\ell_2^n)$ unitary operators with the following spectral decompositions

$$U_i = \sum_{k=1}^{n} \lambda_k^{(i)} P_{\xi_k^{(i)}}, \ i = 1, 2, 3.$$

(See Section 4.2.)

We start with the following approximation lemma.

Lemma 5.27. *Let $U_0, U_1, U_2 \in \mathcal{B}(\ell_2^n)$ be unitary operators and let $(F_m)_m$ be a sequence of unitaries such that $F_m \to U_0$ in the uniform operator topology as $m \to \infty$. Let $\psi \in C(\mathbb{T}^3)$. Then*

$$\Gamma^{F_m, U_1, U_2}(\psi) \longrightarrow \Gamma^{U_0, U_1, U_2}(\psi) \quad \textit{as } m \to \infty.$$

Proof. Let $F \in \mathcal{B}(\ell_2^n)$ be any unitary operator. Consider a spectral decomposition $F = \sum_{i=1}^n \xi_i P_{\xi_i}$. Let $X, Y \in \mathcal{B}(\ell_2^n)$. According to (4.17), we have

$$\begin{aligned}
\Gamma^{F, U_1, U_2}(\psi)(X, Y) &= \sum_{i,j,k=1}^{n} \psi(\xi_i, \xi_k^{(1)}, \xi_j^{(2)}) P_{\xi_i} X P_{\xi_k^{(1)}} Y P_{\xi_j^{(2)}} \\
&= \sum_{j,k=1}^{n} \Big(\sum_{i=1}^{n} \psi(\xi_i, \xi_k^{(1)}, \xi_j^{(2)}) P_{\xi_i} \Big) X P_{\xi_k^{(1)}} Y P_{\xi_j^{(2)}} \\
&= \sum_{j,k=1}^{n} \psi(F, \xi_k^{(1)}, \xi_j^{(2)}) X P_{\xi_k^{(1)}} Y P_{\xi_j^{(2)}},
\end{aligned}$$

where $\psi(F, \xi_k^{(1)}, \xi_j^{(2)})$ is the operator obtained by applying the continuous functional calculus of F to $\psi(\cdot\,, \xi_k^{(1)}, \xi_j^{(2)})$.

For any $\varphi \in C(\mathbb{T})$, the mapping $F \mapsto \varphi(F)$ is continuous from the set of unitaries of $\mathcal{B}(\ell_2^n)$ into $\mathcal{B}(\ell_2^n)$. Hence for any $j, k = 1, \ldots, n$,

$$\psi(F_m, \xi_k^{(1)}, \xi_j^{(2)}) \longrightarrow \psi(U_0, \xi_k^{(1)}, \xi_j^{(2)}) \quad \text{as } m \to \infty.$$

From the above computation we deduce that for any $X, Y \in \mathcal{B}(\ell_2^n)$,

$$\Gamma^{F_m,U_1,U_2}(\psi)(X,Y) \longrightarrow \Gamma^{U_0,U_1,U_2}(\psi)(X,Y) \quad \text{as } m \to \infty.$$

Since $\Gamma^{F_m,U_1,U_2}(\psi)$ and $\Gamma^{U_0,U_1,U_2}(\psi)$ act on a finite dimensional space, this proves the result. □

Remark 5.28. *Similarly for any unitary operators $U_0, U_1 \in \mathcal{B}(\ell_2^n)$, for any sequence $(F_m)_m$ of unitaries on ℓ_2^n such that $F_m \to U_0$ as $m \to \infty$, and for any $\phi \in C(\mathbb{T}^2)$, we have*

$$\Gamma^{F_m,U_1}(\phi) \longrightarrow \Gamma^{U_0,U_1}(\phi) \quad \textit{as } m \to \infty.$$

We now turn to perturbation theory. In Section 5.2 we defined the divided differences for functions defined on $\mathbb{R}$. A similar definition can be given for complex function defined on a $\mathbb{T}$ as follow. Let $f \in C^1(\mathbb{T})$. The divided difference of first order is the function $f^{[1]} \colon \mathbb{T}^2 \to \mathbb{C}$ defined by

$$f^{[1]}(z_0, z_1) := \begin{cases} \frac{f(z_0)-f(z_1)}{z_0-z_1}, & \text{if } z_0 \neq z_1 \\ \frac{d}{dz} f(z)_{|z=z_0} & \text{if } z_0 = z_1 \end{cases}, \qquad z_0, z_1 \in \mathbb{T}.$$

This is a continuous function, symmetric in the two variables (z_0, z_1).

Assume further that $f \in C^2(\mathbb{T})$. Then the divided difference of the second order is the function $f^{[2]} \colon \mathbb{T}^3 \to \mathbb{C}$ defined by

$$f^{[2]}(z_0, z_1, z_2) := \begin{cases} \frac{f^{[1]}(z_0,z_1)-f^{[1]}(z_1,z_2)}{z_0-z_2}, & \text{if } z_0 \neq z_2, \\ \frac{d}{dz} f^{[1]}(z, z_1)_{|z=z_0}, & \text{if } z_0 = z_2 \end{cases}, \qquad z_0, z_1, z_2 \in \mathbb{T}.$$

Note that $f^{[2]}$ is a continuous function, which is symmetric in the three variables (z_0, z_1, z_2).

Let $U_0, U_1 \in \mathcal{B}(\ell_2^n)$ be unitary operators and $f \in C^1(\mathbb{T})$. Then

$$f(U_0) - f(U_1) = \Gamma^{U_0,U_1}(f^{[1]})(U_0 - U_1). \tag{5.43}$$

See [Pel05] and the references therein for a proof of this result. See also [CMPST16a, Subsection 3.4] for an elementary argument.

Let $Z \in \mathcal{B}(\ell_2^n)$ be a self-adjoint operator and let $U \in \mathcal{B}(\ell_2^n)$ be a unitary operator. Then the function $t \mapsto f(e^{itZ}U)$ is differentiable and

$$\frac{d}{dt}\big(f(e^{itZ}U)\big)_{|t=0} = T^{U,U}_{f^{[1]}}(iZU). \tag{5.44}$$

Indeed by (5.43), we have

$$\frac{f(e^{itZ}U) - f(U)}{t} = T^{e^{itZ}U,U}_{f^{[1]}}\Big(\frac{e^{itZ}U - U}{t}\Big)$$

for any $t \neq 0$. Since $\frac{d}{dt}\big(e^{itZ}\big)_{|t=0} = iZ$, the result follows from Remark 5.28.

The following proposition is the unitary version of Corollary 5.6. In the finite dimensional case, we can give an elementary proof.

Proposition 5.29. *Let $f \in C^2(\mathbb{T})$ and let $U_0, U_1, U_2 \in \mathcal{B}(\ell_2^n)$ be unitary operators. Then for all $X \in \mathcal{B}(\ell_2^n)$ we have*

$$\Gamma^{U_0,U_2}(f^{[1]})(X) - \Gamma^{U_1,U_2}(f^{[1]})(X) = \Gamma^{U_0,U_1,U_2}(f^{[2]})(U_0 - U_1, X).$$

We first prove the following lemma.

Lemma 5.30. *Let $U_0, U_1, U_2 \in \mathcal{B}(\mathbb{C}^n)$ be unitary operators. Let I_n be the identity operator in $\mathcal{B}(\mathbb{C}^n)$. Then for $j = 0, 1$ we have*

(i)
$$\Gamma^{U_0,U_1,U_2}(\psi)(U_j, X) = \Gamma^{U_0,U_1,U_2}(\psi_j)(I_n, X), \quad X \in \mathcal{B}(\mathbb{C}^n),$$

where
$$\psi_j(x_0, x_1, x_2) = x_j \psi(x_0, x_1, x_2), \;\; x_0, x_1, x_2 \in \mathbb{R}.$$

(ii)
$$\Gamma^{U_j,U_2}(\phi)(X) = \Gamma^{U_0,U_1,U_2}(\tilde{\psi}_j)(I_n, X), \quad X \in \mathcal{B}(\mathbb{C}^n),$$

where
$$\tilde{\psi}_j(x_0, x_1, x_2) = \phi(x_j, x_2), \;\; x_0, x_1, x_2 \in \mathbb{R}.$$

Proof. Let us prove the assertion for $j = 0$ only. The proof for $j = 1$ is similar.
(i). For $X \in \mathcal{B}(\mathbb{C}^n)$ we have

$$\begin{aligned}
\Gamma^{U_0,U_1,U_2}(\psi)(U_0, X) &= \sum_{i,j,k=1}^{n} \psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) P_{\xi_i^{(0)}} U_0 P_{\xi_k^{(1)}} X P_{\xi_j^{(2)}} \\
&= \sum_{i,j,k=1}^{n} \psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) P_{\xi_i^{(0)}} \Big(\sum_{r=1}^{n} \lambda_r^{(0)} P_{\xi_r^{(0)}} \Big) P_{\xi_k^{(1)}} X P_{\xi_j^{(2)}} \\
&= \sum_{i,j,k=1}^{n} \lambda_i^{(0)} \psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) P_{\xi_i^{(0)}} I_n P_{\xi_k^{(1)}} X P_{\xi_j^{(2)}} \\
&= \Gamma^{U_0,U_1,U_2}(\psi_0)(I_n, X).
\end{aligned}$$

(ii). For $X \in \mathcal{B}(\mathbb{C}^n)$ we have

$$\begin{aligned}
\Gamma^{U_0,U_1,U_2}(\tilde{\psi}_0)(I_n, X) &= \sum_{i,j,k=1}^{n} \tilde{\psi}_0(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) P_{\xi_i^{(0)}} I_n P_{\xi_k^{(1)}} X P_{\xi_j^{(2)}} \\
&= \sum_{i,j=1}^{n} \phi(\lambda_i^{(0)}, \lambda_j^{(2)}) P_{\xi_i^{(0)}} \Big(\sum_{k=1}^{n} P_{\xi_k^{(1)}} \Big) X P_{\xi_j^{(2)}} \\
&= \sum_{i,j=1}^{n} \phi(\lambda_i^{(0)}, \lambda_j^{(2)}) P_{\xi_i^{(0)}} X P_{\xi_j^{(2)}} \\
&= \Gamma^{U_0,U_2}(\phi)(X).
\end{aligned}$$

□

Proof of Proposition 5.29. Let $X \in \mathcal{B}(\mathbb{C}^n)$ and let $\psi = f^{[2]}$ and $\phi = f^{[1]}$. Setting $\psi_0, \psi_1, \tilde{\psi}_0, \tilde{\psi}_1$ as in Lemma 5.30 (i), (ii), we have

$$\begin{aligned}(\psi_0 - \psi_1)(x_0, x_1, x_2) &= x_0 f^{[2]}(x_0, x_1, x_2) - x_1 f^{[2]}(x_0, x_1, x_2) \\ &= f^{[1]}(x_0, x_2) - f^{[1]}(x_1, x_2) \\ &= (\tilde{\psi}_0 - \tilde{\psi}_1)(x_0, x_1, x_2).\end{aligned} \tag{5.45}$$

Therefore, by Lemma 5.30, we obtain

$$\begin{aligned}\Gamma^{U_0,U_1,U_2}(f^{[2]})(U_0 - U_1, X) &= \Gamma^{U_0,U_1,U_2}(f^{[2]})(U_0, X) - \Gamma^{U_0,U_1,U_2}(f^{[2]})(U_1, X) \\ &\overset{Lem5.30(i)}{=} \Gamma^{U_0,U_1,U_2}(\psi_0)(I_n, X) - \Gamma^{U_0,U_1,U_2}(\psi_1)(I_n, X) \\ &= \Gamma^{U_0,U_1,U_2}(\psi_0 - \psi_1)(I_n, X) \\ &\overset{(5.45)}{=} \Gamma^{U_0,U_1,U_2}(\tilde{\psi}_0 - \tilde{\psi}_1)(I_n, X) \\ &= \Gamma^{U_0,U_1,U_2}(\tilde{\psi}_0)(I_n, X) - \Gamma^{U_0,U_1,U_2}(\tilde{\psi}_1)(I_n, X) \\ &\overset{Lem5.30(ii)}{=} \Gamma^{U_0,U_2}(f^{[1]})(X) - \Gamma^{U_1,U_2}(f^{[1]})(X).\end{aligned}$$

□

We conclude this section with a formula relating the second order perturbation operator (5.7) with a combination of operator integrals.

Theorem 5.31. *For any self-adjoint operator $Z \in \mathcal{B}(\ell_2^n)$, for any unitary operator $U \in \mathcal{B}(\ell_2^n)$ and for any $f \in C^2(\mathbb{T})$, we have*

$$\begin{aligned}f(e^{iZ}U) - f(U) - \frac{d}{dt}\big(f(e^{itZ}U)\big)_{|t=0} \\ = \Gamma^{e^{iZ}U,U,U}(f^{[2]})(e^{iZ}U - U, iZU) + \Gamma^{e^{iZ}U,U}(f^{[1]})(e^{iZ}U - U - iZU).\end{aligned} \tag{5.46}$$

Proof. By (5.43) we have

$$f(e^{iZ}U) - f(U) = \Gamma^{e^{iZ}U,U}(f^{[1]})(e^{iZ}U - U).$$

Combining with (5.44), we obtain

$$f(e^{iZ}U) - f(U) - \frac{d}{dt}\big(f(e^{itZ}U)\big)_{|t=0} = \Gamma^{e^{iZ}U,U}(f^{[1]})(e^{iZ}U - U) - \Gamma^{U,U}(f^{[1]})(iZU).$$

By linearity, the right-hand side can be written as

$$\Gamma^{e^{iZ}U,U}(f^{[1]})(e^{iZ}U - U - iZU) + \big(\Gamma^{e^{iZ}U,U}(f^{[1]})(iZU) - \Gamma^{U,U}(f^{[1]})(iZU)\big).$$

Applying Proposition 5.29, we obtain that

$$\Gamma^{e^{iZ}U,U}(f^{[1]})(iZU) - \Gamma^{U,U}(f^{[1]})(iZU) = \Gamma^{e^{iZ}U,U,U}(f^{[2]})(e^{iZ}U - U, iZU),$$

and this yields the desired identity (5.46). □

5.4.2 Finite-dimensional constructions

In this section we establish various estimates concerning finite dimensional operators. The symbol 'const' will stand for uniform positive constants, not depending on the dimension.

The estimates we are going to establish in this section start from a result going back to [AS05]. Let $h\colon [-e^{-1}, e^{-1}] \to \mathbb{R}$ be the function defined by

$$h(x) := \begin{cases} |x|\Big(\log\Big|\log\frac{|x|}{e}\Big|\Big)^{-\frac{1}{2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Then h is a C^1-function. We may extend it to a 2π-periodic C^1-function, that we still denote by h for convenience.

According to [AS05, Section 3], there exist a constant $c > 0$ and, for any $n \geq 3$, self-adjoint operators $R_n, D_n \in \mathcal{B}(\ell^2_{2n})$ such that

$$\|R_nD_n - D_nR_n\|_\infty \leq \pi \tag{5.47}$$

and

$$\big\|R_nh(D_n) - h(D_n)R_n\big\|_\infty \geq c\,\log(n)^{\frac{1}{2}}. \tag{5.48}$$

By changing the dimension from $2n$ to $2n+1$ and adding a zero on the diagonal, one may obtain the above results for some self-adjoint operators $R_n, D_n \in \mathcal{B}(\ell^2_{2n+1})$ satisfying the additional property

$$0 \in \sigma(D_n). \tag{5.49}$$

We shall derive the following result.

Theorem 5.32. *For any $n \geq 3$, there exist self-adjoint operators $A_n, B_n \in \mathcal{B}(\ell^2_{2n+1})$ such that $B_n \neq 0$, $0 \in \sigma(A_n)$,*

$$\big\|h(A_n + B_n) - h(A_n)\big\|_\infty \geq \text{const}\,\log(n)^{\frac{1}{2}}\|B_n\|_\infty,$$

and the operators A_n and $A_n + B_n$ are conjugate. That is, there exists a unitary operator $S_n \in \mathcal{B}(\ell^2_{2n+1})$ such that $A_n + B_n = S_n^{-1}A_nS_n$.

Proof. Let us first observe that for any $N \geq 1$ and any operators $X, Y \in \mathcal{B}(\ell^2_N)$,

$$\frac{e^{itX}Y - Ye^{itX}}{t} \longrightarrow i(XY - YX) \quad \text{as } t \to 0. \tag{5.50}$$

Indeed, this follows from the fact that $\frac{d}{dt}(e^{itX})_{|t=0} = iX$.

Consider D_n and R_n satisfying (5.47), (5.48) and (5.49). For any $t > 0$, define

$$B_{n,t} := e^{itR_n}D_ne^{-itR_n} - D_n.$$

On the one hand, applying (5.50) with $X = R_n$ and $Y = D_n$, we obtain that

$$\begin{aligned}\frac{1}{t}\|B_{n,t}\|_\infty &= \frac{1}{t}\big\|e^{itR_n}D_ne^{-itR_n} - D_n\big\|_\infty \\ &= \frac{1}{t}\big\|e^{itR_n}D_n - D_ne^{itR_n}\big\|_\infty\end{aligned}$$

$$\longrightarrow \|R_nD_n - D_nR_n\|_\infty$$

as $t \to 0$.

On the other hand, using the identity

$$h(e^{itR_n}D_ne^{-itR_n}) = e^{itR_n}h(D_n)e^{-itR_n}$$

and applying (5.50) with $X = R_n$ and $Y = h(D_n)$, we have

$$\begin{aligned}\frac{1}{t}\big\|h(D_n + B_{n,t}) - h(D_n)\big\|_\infty &= \frac{1}{t}\big\|e^{itR_n}h(D_n)e^{-itR_n} - h(D_n)\big\|_\infty \\ &= \frac{1}{t}\big\|e^{itR_n}h(D_n) - h(D_n)e^{itR_n}\big\|_\infty \\ &\longrightarrow \|R_nh(D_n) - h(D_n)R_n\|_\infty\end{aligned}$$

as $t \to 0$.

Therefore, there exists $t > 0$ such that

$$\frac{t}{2}\|R_nD_n - D_nR_n\|_\infty \leq \|B_{n,t}\|_\infty \leq 2\pi t \tag{5.51}$$

and

$$\big\|h(D_n + B_{n,t}) - h(D_n)\big\|_\infty \geq c\,\frac{\log(n)^{\frac{1}{2}}}{2}\,t.$$

The above two estimates lead to

$$\big\|h(D_n + B_{n,t}) - h(D_n)\big\|_\infty \geq \frac{c}{4\pi}\,\log(n)^{\frac{1}{2}}\,\|B_{n,t}\|_\infty.$$

Furthermore property (5.48) implies that D_n and R_n do not commute. Hence the first inequality in (5.51) ensures that $B_{n,t} \neq 0$.

To get the result, we set $A_n = D_n$ and $B_n = B_{n,t}$. According to the definition of $B_{n,t}$, the operators A_n and $A_n + B_n$ are conjugate. All other properties of the statement of the theorem follow from the above estimates and (5.49). □

Let $g \in C^1(\mathbb{T})$ be the unique function satisfying

$$g(e^{i\theta}) = h(\theta), \qquad \theta \in \mathbb{R}. \tag{5.52}$$

The following theorem translates the preceding result into the setting of unitary operators.

Theorem 5.33. *For any $n \geq 3$, there exist unitary operators $H_n, K_n \in \mathcal{B}(\ell^2_{2n+1})$ such that*

$$H_n \neq K_n, \quad \sigma(H_n) = \sigma(K_n), \quad 1 \in \sigma(H_n),$$

and

$$\|g(K_n) - g(H_n)\|_\infty \geq \text{const } \log(n)^{\frac{1}{2}}\|K_n - H_n\|_\infty. \tag{5.53}$$

Proof. Given any $n \geq 3$, let A_n, B_n be the operators from Theorem 5.32, and set

$$H_n = e^{iA_n} \qquad \text{and} \qquad K_n = e^{i(A_n+B_n)}.$$

These are unitary operators. Since A_n and $A_n + B_n$ are conjugate, they have the same spectrum hence in turn, $\sigma(H_n) = \sigma(K_n)$. Moreover $1 \in \sigma(H_n)$ since $0 \in \sigma(A_n)$. Since A_n and $A_n + B_n$ are conjugate but different, their sets of spectral projections are different. This implies that $H_n \neq K_n$.

By construction we have

$$g(H_n) = h(A_n) \qquad \text{and} \qquad g(K_n) = h(A_n + B_n).$$

Therefore, by Theorem 5.32, we have

$$\|g(K_n) - g(H_n)\|_\infty \geq \text{const} \log(n)^{\frac{1}{2}} \|B_n\|_\infty.$$

Moreover

$$\|K_n - H_n\|_\infty = \left\|e^{i(A_n+B_n)} - e^{iA_n}\right\|_\infty \leq \|B_n\|_\infty$$

by [PS11, Lemma 8]. This yields the result. □

Let $f \colon \mathbb{T} \to \mathbb{C}$ be defined by

$$f(z) = (z-1)g(z), \qquad z \in \mathbb{T}. \tag{5.54}$$

It turns out that $f \in C^2(\mathbb{T})$. This follows from the definition of h, which is C^2 on $(-e^{-1}, e^{-1}) \setminus \{0\}$, and the fact that $\lim_{x\to 0} xh''(x) = 0$. Details are left to the reader.

We also define an auxiliary function $\varsigma : \mathbb{T}^3 \to \mathbb{C}$ given by

$$\varsigma(z_0, z_1, z_2) = z_1 f^{[2]}(z_0, z_1, z_2). \tag{5.55}$$

Lemma 5.34. *For any $z_0, z_2 \in \mathbb{T}$, we have*

$$\varsigma(z_0, 1, z_2) = g^{[1]}(z_0, z_2).$$

Proof. By the definition of ς, and since $z_1 = 1$, it is enough to prove that

$$f^{[2]}(z_0, 1, z_2) = g^{[1]}(z_0, z_2), \qquad z_0, z_2 \in \mathbb{T}.$$

We have to consider several different cases. Let us first assume that $z_0 \neq z_2$. If $z_0 \neq 1$ and $z_2 \neq 1$, then we have

$$\begin{aligned} f^{[2]}(z_0, 1, z_2) &= \frac{f^{[1]}(z_0, 1) - f^{[1]}(1, z_2)}{z_0 - z_2} = \frac{\frac{f(z_0)-f(1)}{z_0-1} - \frac{f(1)-f(z_2)}{1-z_2}}{z_0 - z_2} \\ &= \frac{g(z_0) - g(z_2)}{z_0 - z_2} = g^{[1]}(z_0, z_2). \end{aligned}$$

If $z_0 = 1$ and $z_2 \neq 1$, then using $\frac{d}{dz} f(z)_{|z=1} = g(1) = h(0) = 0$, we have

$$\begin{aligned} f^{[2]}(1, 1, z_2) &= \frac{f^{[1]}(1, 1) - f^{[1]}(1, z_2)}{1 - z_2} = \frac{\frac{d}{dz} f(z)_{|z=1} - \frac{f(1)-f(z_2)}{1-z_2}}{1 - z_2} \\ &= \frac{-g(z_2)}{1 - z_2} = g^{[1]}(1, z_2). \end{aligned}$$

The argument is similar, when $z_0 \neq 1$ and $z_2 = 1$.

Assume now that $z_0 = z_2$. Using the fact that $f^{[1]}(z,1) = g(z)$ for any z, we obtain in this case that

$$f^{[2]}(z_0, 1, z_0) = \frac{d}{dz} f^{[1]}(z,1)_{|z=z_0} = \frac{d}{dz} g(z)_{|z=z_0} = g^{[1]}(z_0, z_0).$$

□

Corollary 5.35. *For any $n \geq 3$, there exist unitary operators $H_n, K_n \in \mathcal{B}(\ell^2_{2n+1})$ such that*

$$\sigma(H_n) = \sigma(K_n),$$

and

$$\left\|\Gamma^{K_n,H_n,H_n}(\varsigma) : \mathcal{S}^2_{2n+1} \times \mathcal{S}^2_{2n+1} \to \mathcal{S}^1_{2n+1}\right\| \geq \text{const } \log(n)^{\frac{1}{2}}. \tag{5.56}$$

Proof. Take H_n, K_n as in Theorem 5.33; these unitary operators have the same spectrum. Let $\{\mu_k\}_{k=1}^{2n+1}$ be the sequence of eigenvalues of the operator H_n, counted with multiplicity. Since $1 \in \sigma(H_n)$, we may assume that $\mu_1 = 1$. According to (4.19) and Theorem 3.4, we have

$$\left\|\Gamma^{K_n,H_n,H_n}(\varsigma) : \mathcal{S}^2_{2n+1} \times \mathcal{S}^2_{2n+1} \to \mathcal{S}^1_{2n+1}\right\| = \max_{1\leq k\leq 2n+1} \left\|\Gamma^{K_n,H_n}(\varsigma_k) \colon \mathcal{S}^\infty_{2n+1} \to \mathcal{S}^\infty_{2n+1}\right\|,$$

where, for any $k = 1, \ldots, 2n+1$, we set

$$\varsigma_k(z_0, z_1) := \varsigma(z_0, \mu_k, z_1), \quad z_0, z_1 \in \mathbb{T}.$$

In particular, the inequality

$$\left\|\Gamma^{K_n,H_n,H_n}(\varsigma) \colon \mathcal{S}^2_{2n+1} \times \mathcal{S}^2_{2n+1} \to \mathcal{S}^1_{2n+1}\right\| \geq \left\|\Gamma^{K_n,H_n}(\varsigma_1) \colon \mathcal{S}^\infty_{2n+1} \to \mathcal{S}^\infty_{2n+1}\right\|$$

holds. From Lemma 5.34, we have that

$$\varsigma_1(z_0, z_1) = \varsigma(z_0, 1, z_1) = g^{[1]}(z_0, z_1).$$

Therefore, we obtain

$$\left\|\Gamma^{K_n,H_n,H_n}(\varsigma) \colon \mathcal{S}^2_{2n+1} \times \mathcal{S}^2_{2n+1} \to \mathcal{S}^1_{2n+1}\right\| \geq \left\|\Gamma^{K_n,H_n}(g^{[1]}) \colon \mathcal{S}^\infty_{2n+1} \to \mathcal{S}^\infty_{2n+1}\right\|. \tag{5.57}$$

Since $H_n \neq K_n$, we derive

$$\left\|\Gamma^{K_n,H_n,H_n}(\varsigma) \colon \mathcal{S}^2_{2n+1} \times \mathcal{S}^2_{2n+1} \to \mathcal{S}^1_{2n+1}\right\| \geq \frac{\left\|\Gamma^{K_n,H_n}(g^{[1]})(K_n - H_n)\right\|_\infty}{\|K_n - H_n\|_\infty}.$$

From the identity (5.43), we have $\Gamma^{K_n,H_n}(g^{[1]})(K_n - H_n) = g(K_n) - g(H_n)$. Hence the above inequality means that

$$\left\|\Gamma^{K_n,H_n,H_n}(\varsigma) : \mathcal{S}^2_{2n+1} \times \mathcal{S}^2_{2n+1} \to \mathcal{S}^1_{2n+1}\right\| \geq \frac{\|g(K_n) - g(H_n)\|_\infty}{\|K_n - H_n\|_\infty}.$$

Applying (5.53) we obtain the desired estimate. □

We are now ready to prove the final estimate of this section.

Corollary 5.36. *For any $n \geq 3$, there exist a self-adjoint operator $W_n \in \mathcal{B}(\ell^2_{8n+4})$ with $\|W_n\|_2 \leq 1$ and a unitary operator $U_n \in \mathcal{B}(\ell^2_{8n+4})$ such that*

$$\left\|\Gamma^{U_n,U_n,U_n}(f^{[2]})(W_nU_n, W_nU_n)\right\|_1 \geq \text{const } \log(n)^{\frac{1}{2}}. \tag{5.58}$$

Proof. We take H_n and K_n given by Corollary 5.35. Then we consider

$$V_n := \begin{pmatrix} K_n & 0 \\ 0 & H_n \end{pmatrix} \quad \text{and then} \quad U_n := \begin{pmatrix} V_n & 0 \\ 0 & V_n \end{pmatrix}. \tag{5.59}$$

Then V_n is a unitary operator acting on ℓ^2_{4n+2} and U_n is a unitary operator acting on ℓ^2_{8n+4}.

We claim that there exists a self-adjoint operator $W_n \in \mathcal{B}(\ell^2_{8n+4})$ such that $\|W_n\|_2 \leq 1$ and

$$\left\|\Gamma^{U_n,U_n,U_n}(\varsigma)(W_n, W_n)\right\|_1 \geq \text{const } \log(n)^{\frac{1}{2}}.$$

Indeed, using (5.56) and the fact that H_n and K_n have the same sprectrum, this follows from the proofs of [CMPST16a, Lemmas 22-25]. Indeed the arguments there can be used word for word in the present case. It therefore suffices to show

$$\left\|\Gamma^{U_n,U_n,U_n}(\varsigma)(W_n, W_n)\right\|_1 = \left\|\Gamma^{U_n,U_n,U_n}(f^{[2]})(W_nU_n, W_nU_n)\right\|_1. \tag{5.60}$$

For that purpose we set $N = 8n + 4$ and consider a spectral decomposition $U_n = \sum_{i=1}^N z_iP_i$ of U_n. Then by (4.17) we have

$$\begin{aligned}
\Gamma^{U_n,U_n,U_n}(f^{[2]})(W_nU_n, W_nU_n) &= \sum_{i,j,k=1}^N f^{[2]}(z_i, z_k, z_j)P_i(W_nU_n)P_k(W_nU_n)P_j \\
&= \sum_{i,j,k=1}^N f^{[2]}(z_i, z_k, z_j)P_iW_n\Big(\sum_{l=1}^N z_lP_l\Big)P_kW_nP_jU_n \\
&= \sum_{i,j,k=1}^N z_kf^{[2]}(z_i, z_k, z_j)P_iW_nP_kW_nP_jU_n \\
&\overset{(5.55)}{=} \sum_{i,j,k=1}^N \varsigma(z_i, z_k, z_j)P_iW_nP_kW_nP_jU_n \\
&= \Gamma^{U_n,U_n,U_n}(\varsigma)(W_n, W_n)U_n.
\end{aligned}$$

Since U_n is a unitary, this equality implies (5.60), which completes the proof. ☐

5.4.3 A solution to Peller's problem for unitary operators

In this section, we answer Peller's question raised in [Pel05, Problem 1] in the negative.

Theorem 5.37. *There exist a function $f \in C^2(\mathbb{T})$, a separable Hilbert space $\mathcal{H}$, a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a self-adjoint operator $Z \in \mathcal{S}^2(\mathcal{H})$ such that*

$$f\big(e^{iZ}U\big) - f(U) - \frac{d}{dt}\big(f(e^{itZ}U)\big)_{|t=0} \notin \mathcal{S}^1(\mathcal{H}). \tag{5.61}$$

In the above statement, $\frac{d}{dt}\big(f(e^{itZ}U)\big)_{|t=0}$ denotes the derivative of this function at $t = 0$. We refer to [Pel05, (2.7)] and the references therein for the facts that for any $f \in C^1(\mathbb{T})$, for any unitary operator $U \in \mathcal{B}(\mathcal{H})$ and any self-adjoint operator $Z \in \mathcal{S}^2(\mathcal{H})$, the difference operator $f\big(e^{iZ}U\big) - f(U)$ belongs to $\mathcal{S}^2(\mathcal{H})$ and the function $t \mapsto f(e^{itZ}U)$ is differentiable from $\mathbb{R}$ into $\mathcal{S}^2(\mathcal{H})$. Therefore, the operator in (5.61) belongs to $\mathcal{S}^2(\mathcal{H})$.

Theorem 5.37 will be proved with the function f given by (5.54). We will combine a direct sum argument and the following lemma, whose proof relies on Corollary 5.36.

Lemma 5.38. *For any $n \geq 1$, there exist a non zero self-adjoint operator $Z_n \in \mathcal{B}(\ell^2_{8n+4})$ and a unitary operator $U_n \in \mathcal{B}(\ell^2_{8n+4})$, such that*

$$\sum_{n=1}^{\infty} \|Z_n\|_2^2 < \infty, \tag{5.62}$$

and

$$\lim_{n\to\infty} \frac{\Big\| f(e^{iZ_n}U_n) - f(U_n) - \frac{d}{dt}\big(f(e^{itZ_n}U_n)\big)_{|t=0} \Big\|_1}{\|Z_n\|_2^2} = \infty. \tag{5.63}$$

Proof. We fix $n \geq 3$ and we take W_n and U_n given by Corollary 5.36. Note that changing W_n into $\|W_n\|_2^{-1}W_n$, we may (and do) assume that $\|W_n\|_2 = 1$. We consider the sequence

$$W_{m,n} = \frac{1}{m}W_n, \qquad m \geq 1,$$

and we set

$$R_{m,n} := f(e^{iW_{m,n}}U_n) - f(U_n) - \frac{d}{dt}\big(f(e^{itW_{m,n}}U_n)\big)_{|t=0}.$$

By Theorem 5.31 we have

$$\begin{aligned} m^2 R_{m,n} = {} & \Gamma^{e^{iW_{m,n}}U_n,U_n,U_n}(f^{[2]})\big(m(e^{iW_{m,n}}U_n - U_n), iW_nU_n\big) \\ & + \Gamma^{e^{iW_{m,n}}U_n,U_n}(f^{[1]})\big(m^2(e^{iW_{m,n}}U_n - U_n - iW_{m,n}U_n)\big). \end{aligned} \tag{5.64}$$

Note that

$$m\big(e^{iW_{m,n}} - I_n\big) \longrightarrow iW_n \quad \text{as } m \to \infty.$$

Hence by Lemma 5.27, we have

$$\Gamma^{e^{iW_{m,n}}U_n,U_n,U_n}(f^{[2]})\big(m(e^{iW_{m,n}}U_n - U_n), iW_nU_n\big) \longrightarrow \Gamma^{U_n,U_n,U_n}(f^{[2]})(iW_nU_n, iW_nU_n)$$

as $m \to \infty$. This result and Corollary 5.36 imply that for m large enough, we have

$$\Big\|\Gamma^{e^{iW_{m,n}}U_n,U_n,U_n}(f^{[2]})\big(m(e^{iW_{m,n}}U_n - U_n), iW_nU_n\big)\Big\|_1 \geq \text{const } \log(n)^{\frac{1}{2}}. \tag{5.65}$$

We now turn to the analysis of the second term in the right hand side of (5.64). Since $f \in C^2(\mathbb{T})$, there exists a constant $K > 0$ (only depending on f and not on either n or the operators U_n and $W_{m,n}$) such that

$$\left\|\Gamma^{e^{iW_{m,n}}U_n,U_n}(f^{[1]}) \colon \mathcal{S}^1_{8n+4} \to \mathcal{S}^1_{8n+4}\right\| \leq K.$$

This follows from [BS73] (see also [Pel85]).

Now observe that

$$m^2\left(e^{iW_{m,n}} - I_n - iW_{m,n}\right) \longrightarrow \frac{W_n^2}{2} \quad \text{as } m \to \infty.$$

Hence we have

$$\left\|\Gamma^{e^{iW_{m,n}}U_n,U_n}(f^{[1]})\left(m^2(e^{iW_{m,n}}U_n - U_n - iW_{m,n}U_n)\right)\right\|_1 \leq K\|W_n^2\|_1 = K\|W_n\|_2^2 \tag{5.66}$$

for m large enough.

Combining (5.65) and (5.66), we deduce from the identity (5.64) the existence of an integer $m \geq 1$ for which we have an estimate

$$m^2\|R_{m,n}\|_1 \geq \text{const } \log(n)^{\frac{1}{2}}. \tag{5.67}$$

We may assume that $m \geq n$, which ensures that

$$\|W_{m,n}\|_2 \leq \frac{1}{n}.$$

Then we set $Z_n = W_{m,n}$. The preceding inequality implies that $\sum_n \|Z_n\|_2^2 < \infty$. Since $\|W_n\|_2 = 1$, we have $\|Z_n\|_2 = \frac{1}{m}$ hence the estimate (5.67) yields (5.63). □

Proof of Theorem 5.37. We apply Lemma 5.38 above. We set

$$\beta_n := \left\|f(e^{iZ_n}U_n) - f(U_n) - \frac{d}{dt}\left(f(e^{itZ_n}U_n)\right)_{|t=0}\right\|_1$$

for any $n \geq 1$. Since $\left\{\beta_n\|Z_n\|_2^{-2}\right\}_{n=1}^{\infty}$ is an unbounded sequence, by (5.63), there exists a positive sequence $(\alpha_n)_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty}\alpha_n < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty}\alpha_n\beta_n\|Z_n\|_2^{-2} = \infty. \tag{5.68}$$

Set

$$N_n = \left[\alpha_n\|Z_n\|_2^{-2}\right] + 1,$$

where $[\,\cdot\,]$ denotes the integer part of a real number. We have both

$$N_n\|Z_n\|_2^2 \leq \alpha_n + \|Z_n\|_2^2 \qquad \text{and} \qquad N_n \geq \alpha_n\|Z_n\|_2^{-2}.$$

Hence it follows from (5.68), (5.62) and (5.63) that

$$\sum_{n=1}^{\infty} N_n \|Z_n\|_2^2 < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} N_n \beta_n = \infty.$$

We let $\mathcal{H}_n = \ell^2_{N_n}(\ell^2_{8n+4})$ and we let $\tilde{Z}_n$ (resp. $\tilde{U}_n$) be the element of $\mathcal{B}(\mathcal{H}_n)$ obtained as the direct sum of N_n copies of Z_n (resp. U_n). Then $\tilde{Z}_n$ is a self-adjoint operator and $\|\tilde{Z}_n\|_2^2 = N_n\|Z_n\|_2^2$. Consequently,

$$\sum_{n=1}^{\infty} \|\tilde{Z}_n\|_2^2 < \infty\,. \tag{5.69}$$

Likewise $\tilde{U}_n$ is a unitary operator and we have

$$\begin{aligned}&\Big\|f(e^{i\tilde{Z}_n}\tilde{U}_n) - f(\tilde{U}_n) - \frac{d}{dt}\big(f(e^{it\tilde{Z}_n}\tilde{U}_n)\big)_{|t=0}\Big\|_1 \\ &\qquad = N_n\Big\|f(e^{iZ_n}U_n) - f(U_n) - \frac{d}{dt}\big(f(e^{itZ_n}U_n)\big)_{|t=0}\Big\|_1 = N_n\beta_n.\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \Big\|f(e^{i\tilde{Z}_n}\tilde{U}_n) - f(\tilde{U}_n) - \frac{d}{dt}\big(f(e^{it\tilde{Z}_n}\tilde{U}_n)\big)_{|t=0}\Big\|_1 = \infty.$$

We finally consider the direct sum

$$\mathcal{H} = \overset{2}{\oplus}_{n\geq 1}\mathcal{H}_n.$$

We let Z be the direct sum of the $\tilde{Z}_n$, defined by $Z(\xi) = \{\tilde{Z}_n(\xi_n)\}_{n=1}^{\infty}$ for any $\xi = \{\xi_n\}_{n=1}^{\infty}$ in $\mathcal{H}$. Property (5.69) ensures that Z is well-defined and belongs to $\mathcal{S}^2(\mathcal{H})$, with $\|Z\|_2^2 = \sum_{n=1}^{\infty}\|\tilde{Z}_n\|_2^2$. Likewise we let U be the direct sum of the $\tilde{U}_n$. This is a unitary operator and $\frac{d}{dt}\big(f(e^{itZ}U)\big)_{|t=0}$ is the direct sum of the $\frac{d}{dt}\big(f(e^{it\tilde{Z}_n}\tilde{U}_n)\big)_{|t=0}$. Therefore

$$\begin{aligned}&\Big\|f(e^{iZ}U) - f(U) - \frac{d}{dt}\big(f(e^{itZ}U)\big)_{|t=0}\Big\|_1 \\ &\qquad = \sum_{n=1}^{\infty} \Big\|f(e^{i\tilde{Z}_n}\tilde{U}_n) - f(\tilde{U}_n) - \frac{d}{dt}\big(f(e^{it\tilde{Z}_n}\tilde{U}_n)\big)_{|t=0}\Big\|_1\,.\end{aligned}$$

Since this sum is infinite, we obtain the assertion (5.61). □

Just like for the selfadjoint case, the theorem above has been generalized in [DPT16] where the authors consructed a counterexample for a $n-$th order version of Peller's problem.

5.5 Perspectives

in Section 5.2, we studied the differentiability in $S^2(\mathcal{H})$. Taking into account the discussion in Section 4.5, it is interesting to study the $\mathcal{S}^p-$differentiability of the mapping

$$t \in \mathbb{R} \mapsto f(A+tB) - f(A)$$

when A and B are selfadjoint operators with $B \in \mathcal{S}^p$ and $f \in C^n(\mathbb{R})$ with possibly further assumptions such as the boundedness of its derivatives. If the results are positive, one can hope to obtain a formula for the Taylor remainder like in Theorem 5.1. We refer to [EKS12] for some existing results about the $\mathcal{S}^p-$differentiability.

In Section 5.4 we gave some formulas for the differentiability in the case of unitary operators in the finite-dimensional case. The results obtained in Section 5.2 for selfadjoint operators can be also studied in the case of unitary operators. Namely, if U is a unitary operator on some Hilbert space $\mathcal{H}$ and if $Z \in \mathcal{S}^2(\mathcal{H})$ is selfadjoint, then one can study the differentiability of

$$t \in \mathbb{R} \mapsto f(e^{itZ}U) - f(U) \in \mathcal{S}^2(\mathcal{H})$$

for $f \in C^n(\mathbb{T})$. We refer e.g. to [DPT16] for some results in this direction.

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