Norms of Schur multipliers and applications

Clément Coine
Central South University

Let $m = [m_{ij}]$ and $a = [a_{ij}]$ be two matrices in $M_n(\mathbb{C})$. The Schur product of m and a, denoted by $m \circ a$, is the entrywise product $m \circ a = [m_{ij}a_{ij}]$. For a fixed matrix m, we can consider the linear mapping $T_m : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by $T_m(a) = m \circ a$. We say that T_m is a Schur multiplier. These simple transformations are applied, for instance, in harmonic analysis and in perturbation theory.

In this mini-course, we will give a few answers to the following question: for which $m \in M_n(\mathbb{C})$ is there a constant C not depending on the dimension n such that we have the inequality

$$\forall a \in M_n(\mathbb{C}), \|m \circ a\| \leq C\|a\|?$$

Depending on the norm on the space of matrices, this question can be simple or highly complicated, and sometimes has no answer (yet). For instance, for the usual operator norm on $M_n(\mathbb{C})$, the solution to this question is a deep result due to A. Grothendieck. We will give several examples of Schur multipliers: given a matrix $a = [a_{ij}]$, how can we compare the norm of a with the norm of the matrices

$$\left[\frac{a_{ij}}{i+j}\right], \begin{bmatrix} a_{11} & & & (0) \\ & a_{22} & & \\ & & \ddots & \\ (0) & & & a_{nn} \end{bmatrix} \text{ or } \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}?$$

Some applications of Schur multipliers in perturbation theory will be provided. Given a continuous function defined on $\mathbb{R}$ and a hermitian matrix $a \in M_n(\mathbb{C})$, we can define the matrix $f(a)$. It is natural to ask whether there exists a constant K (not depending on n) such that, for all hermitian matrices a, b, we have the estimate

$$\|f(a) - f(b)\| \leq K\|a - b\|.$$

In particular, can we, like in real analysis, have such an inequality with $K = \|f'\|$? To study this question we will give a fundamental formula which states that the difference $f(a) - f(b)$ can be expressed as the Schur product of $a - b$ and a certain matrix m. This formula also allows us to give a representation for the derivative of the map $t \in \mathbb{R} \mapsto f(a + tb) \in M_n(\mathbb{C})$ as a certain Schur multiplier.

At last, we will discuss the infinite-dimensional case and give some recent results in this area.