

# APPLICATION OF SCHUR MULTIPLIERS TO A PROBLEM OF PERTURBATION

Clément Coine

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**THEOREM (D. POTAPOV, F. SUKOCHEV, 2011)**

*Let  $1 < p < \infty$ . There exists a constant  $c_p > 0$  such that for any Lipschitz function  $f : \mathbb{R} \rightarrow \mathbb{C}$  and for all  $A, B$  self-adjoints with  $B \in S_p(H)$ ,*

$$\|f(A + B) - f(A)\|_p \leq c_p \|f\|_{Lip_1} \|B\|_p.$$

Let  $A, B$  be self-adjoints operators and  $f \in \mathcal{C}^2(\mathbb{R})$  such that  $\|f''\|_\infty < \infty$ .  
Write

$$\Gamma(A, B, f) = f(A + B) - f(A) - \frac{d}{dt}(f(A + tB))\Big|_{t=0}.$$



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If  $A$  is bounded and  $B \in S^2(\mathcal{H})$ , then

$$f(A + B) - f(A) \quad \text{et} \quad \left. \frac{d}{dt}(f(A + tB)) \right|_{t=0}$$

are well defined and belong to  $S^2(\mathcal{H})$ .

If  $A$  is unbounded. Take  $f : x \mapsto x^2$ . We have

$$f(A + B) - f(A) = AB + BA + B^2 \quad \text{et} \quad \left. \frac{d}{dt}(f(A + tB)) \right|_{t=0} = AB + BA.$$

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### THEOREM (V. PELLER, 2004)

*There exists  $C > 0$  such that for all  $f \in B_{\infty 1}^2$ ,*

$$\|\Gamma(A, B, f)\|_1 \leq C \|f\|_{B_{\infty 1}^2} \|B\|_2^2.$$

Let  $f \in C^2(\mathbb{R})$  be such that  $f(x) = x|x|$  for  $|x| \geq 1$  and  $f^{(j)}(0) = 0$ ,  $j = 0, 1, 2$ .

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THEOREM (C., LE MERDY, POTAPOV, SUKOCHEV, TOMSKOVA, 2015)

*There exist an unbounded self-adjoint operator  $A$  and a self-adjoint operator  $B \in \mathcal{S}^2(\ell^2)$  such that*

$$\Gamma(A, B, f) = f(A + B) - f(A) - \frac{d}{dt}(f(A + tB))\Big|_{t=0} \notin \mathcal{S}^1.$$

$A$  and  $B$  will be of the form  $A = \bigoplus_{n \geq 1} A_n$ ,  $B = \bigoplus_{n \geq 1} B_n$  where  $A_n, B_n \in \mathcal{B}(\mathcal{H}_n)$  with  $\mathcal{H}_n$  finite dimensional.



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We will then have

$$\Gamma(A, B, f) = \bigoplus_{n \geq 1} \left( f(A_n + B_n) - f(A_n) - \frac{d}{dt}(f(A_n + tB_n)) \Big|_{t=0} \right).$$

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### LEMMA

There exist  $C > 0$  and two sequences of operators  $A_n, B_n \in B(\mathbb{C}^{8n+4})$  such that  $\|B_n\|_2^2 \leq \frac{1}{n \log^{3/2} n}$ , for all  $n \geq N$ , and

$$\|f(A_n + B_n) - f(A_n) - \frac{d}{dt}(f(A_n + tB_n)) \Big|_{t=0}\|_1 \geq \frac{C}{n \log^{1/2}(n)}.$$

Soit  $1 \leq p \leq \infty$ .

- A matrix  $N = \{n_{ij}\}_{i,j \geq 1}$  with entries in  $\mathbb{C}$  is called a Schur multiplier on  $S^p$  if the following action

$$S_N(A) := \sum_{i,j \geq 1} n_{ij} a_{ij} E_{ij}, \quad A = \{a_{ij}\}_{i,j \geq 1} \in S^p,$$

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- A three-dimensional matrix  $M = \{m_{ikj}\}_{i,k,j \geq 1}$  with entries in  $\mathbb{C}$  is called a bilinear Schur multiplier onto  $S^p$  if the following action

$$T_M(A, B) := \sum_{i,j,k \geq 1} m_{ikj} a_{ik} b_{kj} E_{ij}, \quad A = \{a_{ij}\}_{i,j \geq 1}, B = \{b_{ij}\}_{i,j \geq 1} \in S^2,$$

defines a bounded bilinear mapping from  $S^2 \times S^2$  onto  $S^p(\ell^2)$ .

**Remarks :** • There is a description of linear Schur multipliers onto  $S^\infty$  (Grothendieck's theorem).

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• If  $N = \{n_{ij}\}_{i,j \geq 1}$  is a linear Schur multipliers onto  $S^p$  then  $\sup_{i,j \geq 1} |n_{ij}| < \infty$ . For  $p = 2$ , this condition is sufficient and we have,

$$\|S_N : \mathcal{S}^2 \rightarrow \mathcal{S}^2\| = \sup_{i,j \geq 1} |n_{ij}|.$$

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### PROPOSITION

Let  $n \in \mathbb{N}$ . Let  $M = \{m_{ikj}\}_{i,k,j=1}^n$  be a three-dimensional matrix and for all  $1 \leq k \leq n$ , denote  $T(k)$  the matrix given by  $T(k) = \{m_{ikj}\}_{i,j=1}^n$ . Then

$$\|T_M : S_n^2 \times S_n^2 \rightarrow S_n^1\| = \max_{1 \leq k \leq n} \|S_{T(k)} : M_n \rightarrow M_n\|.$$



Let  $A_0, A_1 \in B(\mathbb{C}^n)$  be diagonalizable self-adjoint operators. For  $j = 0, 1$ , let  $\xi_j = \{\xi_i^{(j)}\}_{i=1}^n$  be an orthonormal basis of eigenvectors for  $A_j$ , and let  $\{\lambda_i^{(j)}\}_{i=1}^n$  be the associated  $n$ -tuple of eigenvalues.

Denote  $P_{\xi_i^{(j)}} = \xi_i^{(j)} \otimes \overline{\xi_i^{(j)}}$  the projection onto  $\mathbb{C}\xi_i^{(j)}$ .

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Let  $\psi : \mathbb{R}^2 \rightarrow \mathbb{C}$  be a bounded Borel function. Define a linear operator  $T_\phi^{A_0, A_1} : B(\mathbb{C}^n) \rightarrow B(\mathbb{C}^n)$  given by

$$T_\phi^{A_0, A_1}(X) = \sum_{i,k=1}^n \phi(\lambda_i^{(0)}, \lambda_k^{(1)}) P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}}, \quad X \in B(\mathbb{C}^n).$$

Let  $A_0, A_1, A_2 \in B(\mathbb{C}^n)$  be diagonalizable self-adjoint operators and for any  $j = 0, 1, 2$ , let  $\xi_j = \{\xi_i^{(j)}\}_{i=1}^n$  be an orthonormal basis of eigenvectors of  $A_j$  and let  $\{\lambda_i^{(j)}\}_{i=1}^n$  be the corresponding  $n$ -tuple of eigenvalues.

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Let  $\psi : \mathbb{R}^3 \rightarrow \mathbb{C}$  be a bounded Borel function. We define the bilinear mapping

$$T_{\psi}^{A_0, A_1, A_2}(X, Y) = \sum_{i, j, k=1}^n \psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}} Y P_{\xi_j^{(2)}}$$

for any  $X, Y \in B(\mathbb{C}^n)$ .

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$$\|T_{\phi}^{A_0, A_1} : \mathcal{S}_n^{\infty} \rightarrow \mathcal{S}_n^{\infty}\| = \|\{\phi(\lambda_i^{(0)}, \lambda_k^{(1)})\}_{i,k=1}^n : \mathcal{S}_n^{\infty} \rightarrow \mathcal{S}_n^{\infty}\|.$$

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### PROPOSITION

$$\|T_{\psi}^{A_0, A_1, A_2} : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \rightarrow \mathcal{S}_n^1\| = \|\{\psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)})\}_{i,j,k=1}^n : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \rightarrow \mathcal{S}_n^1\|$$



- Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function and assume that  $f$  admits right and left derivatives  $f'_r(x)$  and  $f'_l(x)$  at each  $x \in \mathbb{R}$ .  
The divided difference of first order is defined by

$$f^{[1]}(x_0, x_1) := \begin{cases} \frac{f(x_0) - f(x_1)}{x_0 - x_1}, & \text{if } x_0 \neq x_1 \\ \frac{f'_r(x_0) + f'_l(x_0)}{2} & \text{if } x_0 = x_1 \end{cases}, \quad x_0, x_1 \in \mathbb{R}.$$

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- If  $f \in C^2(\mathbb{R})$ , the divided difference of second order is defined by

$$f^{[2]}(x_0, x_1, x_2) := \begin{cases} \frac{f^{[1]}(x_0, x_1) - f^{[1]}(x_1, x_2)}{x_0 - x_2}, & \text{if } x_0 \neq x_2 \\ \frac{d}{dx_0} f^{[1]}(x_0, x_1), & \text{if } x_0 = x_2 \end{cases}, \quad x_0, x_1, x_2 \in \mathbb{R}.$$

## PROPOSITION

For all self-adjoint  $A_0, A_1 \in B(\mathbb{C}^n)$

$$f(A_0) - f(A_1) = T_{f^{[1]}}^{A_0, A_1}(A_0 - A_1).$$

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If  $f \in C^1(\mathbb{R})$ , the function  $t \mapsto f(A_0 + tA_1)$  is differentiable and

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## THEOREM

For all self-adjoints  $A, B \in B(\mathbb{C}^n)$  and all function  $f \in C^2(\mathbb{R})$ , we have

$$f(A + B) - f(A) - \left. \frac{d}{dt} (f(A + tB)) \right|_{t=0} = T_{f^{[2]}}^{A+B, A, A}(B, B).$$

Define  $f_0(x) = |x|$ .

THEOREM (DAVIES, 1988)

*There exists a constant  $C > 0$  such that for any  $n \geq 1$ , there exist  $A_n, B_n \in \mathcal{B}(\mathbb{C}^{2n})$  self-adjoint such that  $B_n \neq 0$  and*

$$\|f_0(A_n + B_n) - f_0(A_n)\|_1 \geq C \log n \|B_n\|_1.$$

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Since  $f_0(A_n + B_n) - f_0(A_n) = T_{f_0^{[1]}}^{A_n+B_n, A_n}(B_n)$ , this implies that

$$\begin{aligned} \left\| T_{f_0^{[1]}}^{A_n+B_n, A_n} : \mathcal{S}_{2n+1}^1 \rightarrow \mathcal{S}_{2n+1}^1 \right\| &= \left\| T_{f_0^{[1]}}^{A_n+B_n, A_n} : \mathcal{S}_{2n+1}^\infty \rightarrow \mathcal{S}_{2n+1}^\infty \right\| \\ &\geq C \log n. \end{aligned}$$

Let  $f \in C^2(\mathbb{R})$  be such that  $f(x) = x|x|$  for  $|x| \geq 1$  and  $f^{(j)}(0) = 0$ ,  $j = 0, 1, 2$ .



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By the equalities

$$\begin{aligned}
 & \left\| T_{f^{[2]}}^{A_n+B_n, A_n, A_n} : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \rightarrow \mathcal{S}_n^1 \right\| \\
 &= \left\| \{f^{[2]}(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)})\}_{i,j,k=1}^n : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \rightarrow \mathcal{S}_n^1 \right\| \\
 &= \max_{1 \leq k \leq n} \left\| \{f^{[1]}(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)})\}_{i,k=1}^n : M_n \rightarrow M_n \right\|. \\
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and (lots of) technicalities, we obtain the estimate in the finite dimensional case.

Thanks for your attention !