# Application of Schur multipliers to a problem of perturbation

Clément Coine
University of Franche-Comté

20th Conference of the International Linear Algebra Society,
Leuven, Belgium
July 15, 2016

Joint work with Christian Le Merdy, Denis Potapov, Fedor Sukochev and Anna Tomskova

Let $H$ be a Hilbert space and let $T \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the set of compact operators on $H$.

Let $H$ be a Hilbert space and let $T \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the set of compact operators on $H$. Then $T^*T$ is compact, and so is $|T| = \sqrt{T^*T}$.

Let $H$ be a Hilbert space and let $T \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the set of compact operators on $H$. Then $T^*T$ is compact, and so is $|T| = \sqrt{T^*T}$. Moreover, $|T|$ is self-adjoint.

Let $H$ be a Hilbert space and let $T \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the set of compact operators on $H$. Then $T^*T$ is compact, and so is $|T| = \sqrt{T^*T}$. Moreover, $|T|$ is self-adjoint.

There exists then a decreasing sequence of non-negative numbers $(\lambda_n(T))_n$ and an orthonormal family $(u_n)_n \in H$ such that

$$|T| = \sum_n \lambda_n \langle ., u_n \rangle u_n.$$

Let $H$ be a Hilbert space and let $T \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the set of compact operators on $H$. Then $T^*T$ is compact, and so is $|T| = \sqrt{T^*T}$. Moreover, $|T|$ is self-adjoint.
There exists then a decreasing sequence of non-negative numbers $(\lambda_n(T))_n$ and an orthonormal family $(u_n)_n \in H$ such that

$$|T| = \sum_n \lambda_n \langle ., u_n \rangle u_n.$$

By polar decomposition, there exists a partial isometry $U \in \mathcal{B}(H)$ such that $T = U|T|$.

Let $H$ be a Hilbert space and let $T \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the set of compact operators on $H$. Then $T^*T$ is compact, and so is $|T| = \sqrt{T^*T}$. Moreover, $|T|$ is self-adjoint.
There exists then a decreasing sequence of non-negative numbers $(\lambda_n(T))_n$ and an orthonormal family $(u_n)_n \in H$ such that

$$|T| = \sum_n \lambda_n \langle ., u_n \rangle u_n.$$

By polar decomposition, there exists a partial isometry $U \in \mathcal{B}(H)$ such that $T = U|T|$.
Setting $v_n = U(u_n)$, we obtain an orthonormal family $(v_n)_n$ such that

$$T = \sum_n \lambda_n \langle ., u_n \rangle v_n.$$

### Definition

*We define, for* $1 \leq p < \infty$,

$$S^p(H) = \{T \in \mathcal{K}(H) \mid (\lambda_n(T))_n \in \ell_p < \infty\}.$$

If $T \in S_p(H)$, we set $\|T\|_p = \|(\lambda_n(T))_n\|_{\ell_p}$. Notice that

$$\|T\|_= \mathrm{tr}(|T|^p)^{1/p}.$$

### Definition

*We define, for* $1 \leq p < \infty$,

$$S^p(H) = \{T \in \mathcal{K}(H) \mid (\lambda_n(T))_n \in \ell_p < \infty\}.$$

If $T \in S_p(H)$, we set $\|T\|_p = \|(\lambda_n(T))_n\|_{\ell_p}$. Notice that

$$\|T\|_= \mathrm{tr}(|T|^p)^{1/p}.$$

### Theorem

$(S_p(H), \|.\|_p)$ *est un espace de Banach.*

Let $\mathcal{H}$ be a Hilbert space. Let $A, B$ be self-adjoint on $\mathcal{H}$.

Let $\mathcal{H}$ be a Hilbert space. Let $A, B$ be self-adjoint on $\mathcal{H}$.
If $f$ is Lipschitz on $\mathbb{R}$ and $B \in S^1(\mathcal{H})$, do we have

$$f(A+B) - f(A) \in S^1(\mathcal{H})?$$

Let $\mathcal{H}$ be a Hilbert space. Let $A, B$ be self-adjoint on $\mathcal{H}$.
If $f$ is Lipschitz on $\mathbb{R}$ and $B \in S^1(\mathcal{H})$, do we have

$$f(A+B) - f(A) \in S^1(\mathcal{H})?$$

Counter-example by Farforovskaya (1972) : there exist $f \in \mathcal{C}^1(\mathbb{R})$, $A, B$ bounded self-adjoints operators with $B \in S^1(\mathcal{H})$ such that

$$f(A+B) - f(A) \notin S^1(\mathcal{H}).$$

Let $\mathcal{H}$ be a Hilbert space. Let $A, B$ be self-adjoint on $\mathcal{H}$.
If $f$ is Lipschitz on $\mathbb{R}$ and $B \in S^1(\mathcal{H})$, do we have

$$f(A+B) - f(A) \in S^1(\mathcal{H})?$$

Counter-example by Farforovskaya (1972) : there exist $f \in \mathcal{C}^1(\mathbb{R})$, $A, B$ bounded self-adjoints operators with $B \in S^1(\mathcal{H})$ such that

$$f(A+B) - f(A) \notin S^1(\mathcal{H}).$$

V. Peller proved that the result is true when $f$ to certain classes of functions (1985).

Let $\mathcal{H}$ be a Hilbert space. Let $A, B$ be self-adjoint on $\mathcal{H}$.
If $f$ is Lipschitz on $\mathbb{R}$ and $B \in S^1(\mathcal{H})$, do we have

$$f(A+B) - f(A) \in S^1(\mathcal{H})?$$

Counter-example by Farforovskaya (1972) : there exist $f \in \mathcal{C}^1(\mathbb{R})$, $A, B$ bounded self-adjoints operators with $B \in S^1(\mathcal{H})$ such that

$$f(A+B) - f(A) \notin S^1(\mathcal{H}).$$

V. Peller proved that the result is true when $f$ to certain classes of functions (1985).

**Theorem (D. Potapov, F. Sukochev, 2011)**

*Let $1 < p < \infty$. There exists a constant $c_p > 0$ such that for any Lipschitz function $f : \mathbb{R} \to \mathbb{C}$ and for all $A, B$ self-adjoints with $B \in S_p(H)$,*

$$\|f(A+B) - f(A)\|_p \leq c_p \|f\|_{Lip_1} \|B\|_p.$$

Let $A, B$ be self-adjoint operators and $f \in \mathcal{C}^2(\mathbb{R})$ such that $\|f''\|_\infty < \infty$. Write

$$\Gamma(A, B, f) = f(A + B) - f(A) - \frac{d}{dt}(f(A + tB))\big|_{t=0}.$$

Let $A, B$ be self-adjoint operators and $f \in \mathcal{C}^2(\mathbb{R})$ such that $\|f''\|_\infty < \infty$. Write

$$\Gamma(A, B, f) = f(A+B) - f(A) - \frac{d}{dt}(f(A+tB))\big|_{t=0}.$$

Considered for the first time by Koplienko (1984).

Let $A, B$ be self-adjoint operators and $f \in \mathcal{C}^2(\mathbb{R})$ such that $\|f''\|_\infty < \infty$. Write

$$\Gamma(A, B, f) = f(A+B) - f(A) - \frac{d}{dt}(f(A+tB))\big|_{t=0}.$$

Considered for the first time by Koplienko (1984).
If $A$ is bounded and $B \in S^2(\mathcal{H})$, then

$$f(A+B) - f(A) \text{ et } \frac{d}{dt}(f(A+tB))\big|_{t=0}$$

are well defined and belong to $S^2(\mathcal{H})$.

If $A$ is unbounded. Take $f : x \mapsto x^2$. We have

$$f(A+B) - f(A) = AB + BA + B^2 \quad \text{and} \quad \frac{d}{dt}(f(A+tB))\Big|_{t=0} = AB + BA.$$

If $A$ is unbounded. Take $f : x \mapsto x^2$. We have

$$f(A+B) - f(A) = AB + BA + B^2 \quad \text{and} \quad \frac{d}{dt}(f(A+tB))\Big|_{t=0} = AB + BA.$$

In the case when $A$ is unbounded, we define $\Gamma(A, B, f)$ by an approximation process.

If $A$ is unbounded. Take $f : x \mapsto x^2$. We have

$$f(A+B) - f(A) = AB + BA + B^2 \quad \text{and} \quad \frac{d}{dt}(f(A+tB))\Big|_{t=0} = AB + BA.$$

In the case when $A$ is unbounded, we define $\Gamma(A, B, f)$ by an approximation process.

**Theorem (V. Peller, 2004)**

*There exists $C > 0$ such that for all $f \in B^2_{\infty 1}$,*

$$\|\Gamma(A, B, f)\|_1 \leq C\|f\|_{B^2_{\infty 1}} \|B\|_2^2.$$

Let $f \in C^2(\mathbb{R})$ be such that $f(x) = x|x|$ for $|x| \geq 1$ and $f^{(j)}(0) = 0$, $j = 0, 1, 2$.

Let $f \in C^2(\mathbb{R})$ be such that $f(x) = x|x|$ for $|x| \geq 1$ and $f^{(j)}(0) = 0$, $j = 0, 1, 2$.

**Theorem (C., Le Merdy, Potapov, Sukochev, Tomskova, 2015)**

*There exist an unbounded self-adjoint operator $A$ and a self-adjoint operator $B \in \mathcal{S}^2(\ell^2)$ such that*

$$\Gamma(A, B, f) = f(A + B) - f(A) - \frac{d}{dt}(f(A + tB))\Big|_{t=0} \notin \mathcal{S}^1.$$

$A$ and $B$ will be of the form $A = \oplus_{n \geq 1} A_n$, $B = \oplus_{n \geq 1} B_n$ where $A_n, B_n \in \mathcal{B}(\mathcal{H}_n)$ with $\mathcal{H}_n$ finite dimensional.

$A$ and $B$ will be of the form $A = \oplus_{n\geq 1} A_n$, $B = \oplus_{n\geq 1} B_n$ where $A_n, B_n \in \mathcal{B}(\mathcal{H}_n)$ with $\mathcal{H}_n$ finite dimensional.
We will then have

$$\Gamma(A, B, f) = \bigoplus_{n\geq 1} \left( f(A_n + B_n) - f(A_n) - \frac{d}{dt}(f(A_n + tB_n))\Big|_{t=0} \right).$$

$A$ and $B$ will be of the form $A = \oplus_{n\geq 1} A_n$, $B = \oplus_{n\geq 1} B_n$ where $A_n, B_n \in \mathcal{B}(\mathcal{H}_n)$ with $\mathcal{H}_n$ finite dimensional.
We will then have

$$\Gamma(A, B, f) = \bigoplus_{n\geq 1} \left( f(A_n + B_n) - f(A_n) - \frac{d}{dt}(f(A_n + tB_n))\Big|_{t=0} \right).$$

**Lemma**

*There exist $C > 0$ and two sequences of operators $A_n, B_n \in B(\mathbb{C}^{8n+4})$ such that $\|B_n\|_2^2 \leq \frac{1}{n \ \log^{3/2} n}$, for all $n \geq N$, and*

$$\|f(A_n + B_n) - f(A_n) - \frac{d}{dt}(f(A_n + tB_n))\Big|_{t=0}\|_1 \geq \frac{C}{n \log^{1/2}(n)}.$$

Soit $1 \leq p \leq \infty$.

• A matrix $N = \{n_{ij}\}_{i,j\geq 1}$ with entries in $\mathbb{C}$ is called a Schur multiplier on $S^p$ if the following action

$$S_N(A) := \sum_{i,j\geq 1} n_{ij} a_{ij} E_{ij}, \quad A = \{a_{ij}\}_{i,j\geq 1} \in \mathcal{S}^p,$$

defines a bounded operator on $S^p(\ell^2)$.

Soit $1 \leq p \leq \infty$.

• A matrix $N = \{n_{ij}\}_{i,j\geq 1}$ with entries in $\mathbb{C}$ is called a Schur multiplier on $S^p$ if the following action

$$S_N(A) := \sum_{i,j\geq 1} n_{ij} a_{ij} E_{ij}, \quad A = \{a_{ij}\}_{i,j\geq 1} \in \mathcal{S}^p,$$

defines a bounded operator on $S^p(\ell^2)$.

• A three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ with entries in $\mathbb{C}$ is called a bilinear Schur multiplier onto $S^p$ if the following action

$$T_M(A,B) := \sum_{i,j,k\geq 1} m_{ikj} a_{ik} b_{kj} E_{ij}, \quad A = \{a_{ij}\}_{i,j\geq 1}, B = \{b_{ij}\}_{i,j\geq 1} \in \mathcal{S}^2,$$

defines a bounded bilinear mapping from $S^2 \times S^2$ onto $S^p(\ell^2)$.

**Remarks :** • There is a description of linear Schur multipliers on $\mathcal{B}(\ell^2)$ (Grothendieck's theorem).

**Remarks :** • There is a description of linear Schur multipliers on $\mathcal{B}(\ell^2)$ (Grothendieck's theorem).

• If $N = \{n_{ij}\}_{i,j\geq 1}$ is a linear Schur multipliers on $\mathcal{S}^p$ then $\sup_{i,j\geq 1} |n_{ij}| < \infty$. For $p = 2$, this condition is sufficient and we have,

$$\|S_N : \mathcal{S}^2 \to \mathcal{S}^2\| = \sup_{i,j\geq 1} |n_{ij}|.$$

**Remarks :** • There is a description of linear Schur multipliers on $\mathcal{B}(\ell^2)$ (Grothendieck's theorem).

• If $N = \{n_{ij}\}_{i,j\geq 1}$ is a linear Schur multipliers on $\mathcal{S}^p$ then $\sup_{i,j\geq 1} |n_{ij}| < \infty$. For $p = 2$, this condition is sufficient and we have,

$$\|S_N : \mathcal{S}^2 \to \mathcal{S}^2\| = \sup_{i,j\geq 1} |n_{ij}|.$$

• A matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ is a bilinear Schur multiplier onto $\mathcal{S}^2$ if and only if $\sup_{i,j,k\geq 1} |m_{ikj}| < \infty$.

**Remarks :** • There is a description of linear Schur multipliers on $\mathcal{B}(\ell^2)$ (Grothendieck's theorem).

• If $N = \{n_{ij}\}_{i,j\geq 1}$ is a linear Schur multipliers on $\mathcal{S}^p$ then $\sup_{i,j\geq 1} |n_{ij}| < \infty$. For $p = 2$, this condition is sufficient and we have,

$$\|S_N : \mathcal{S}^2 \to \mathcal{S}^2\| = \sup_{i,j\geq 1} |n_{ij}|.$$

• A matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ is a bilinear Schur multiplier onto $\mathcal{S}^2$ if and only if $\sup_{i,j,k\geq 1} |m_{ikj}| < \infty$.

**Proposition**

*Let $n \in \mathbb{N}$. Let $M = \{m_{ikj}\}_{i,k,j=1}^n$ be a three-dimensional matrix and for all $1 \leq k \leq n$, denote $T(k)$ the matrix given by $T(k) = \{m_{ikj}\}_{i,j=1}^n$. Then*

$$\left\| T_M : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1 \right\| = \max_{1\leq k\leq n} \left\| S_{T(k)} : M_n \to M_n \right\|.$$

Let $A_0, A_1 \in B(\mathbb{C}^n)$ be diagonalizable self-adjoint operators. For $j = 0, 1$, let $\xi_j = \{\xi_i^{(j)}\}_{i=1}^n$ be an orthonormal basis of eigenvectors for $A_j$, and let $\{\lambda_i^{(j)}\}_{i=1}^n$ be the associated $n$-tuple of eigenvalues.
Denote $P_{\xi_i^{(j)}} = \xi_i^{(j)} \otimes \overline{\xi_i^{(j)}}$ the projection onto $\mathbb{C}\xi_i^{(j)}$.

Let $A_0, A_1 \in B(\mathbb{C}^n)$ be diagonalizable self-adjoint operators. For $j = 0, 1$, let $\xi_j = \{\xi_i^{(j)}\}_{i=1}^n$ be an orthonormal basis of eigenvectors for $A_j$, and let $\{\lambda_i^{(j)}\}_{i=1}^n$ be the associated $n$-tuple of eigenvalues.
Denote $P_{\xi_i^{(j)}} = \xi_i^{(j)} \otimes \overline{\xi_i^{(j)}}$ the projection onto $\mathbb{C}\xi_i^{(j)}$.

Let $\phi : \mathbb{R}^2 \to \mathbb{C}$ be a bounded Borel function. Define a linear operator $T_\phi^{A_0,A_1} : B(\mathbb{C}^n) \to B(\mathbb{C}^n)$ given by

$$T_\phi^{A_0,A_1}(X) = \sum_{i,k=1}^n \phi(\lambda_i^{(0)}, \lambda_k^{(1)}) P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}}, \quad X \in B(\mathbb{C}^n).$$

Let $A_0, A_1, A_2 \in B(\mathbb{C}^n)$ be diagonalizable self-adjoint operators and for any $j = 0, 1, 2$, let $\xi_j = \{\xi_i^{(j)}\}_{i=1}^n$ be an orthonormal basis of eigenvectors of $A_j$ and let $\{\lambda_i^{(j)}\}_{i=1}^n$ be the corresponding $n$-tuple of eigenvalues.

Let $A_0, A_1, A_2 \in B(\mathbb{C}^n)$ be diagonalizable self-adjoint operators and for any $j = 0, 1, 2$, let $\xi_j = \{\xi_i^{(j)}\}_{i=1}^n$ be an orthonormal basis of eigenvectors of $A_j$ and let $\{\lambda_i^{(j)}\}_{i=1}^n$ be the corresponding $n$-tuple of eigenvalues.

Let $\psi : \mathbb{R}^3 \to \mathbb{C}$ be a bounded Borel function. We define the bilinear mapping

$$T_\psi^{A_0,A_1,A_2}(X, Y) = \sum_{i,j,k=1}^{n} \psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}} Y P_{\xi_j^{(2)}}$$

for any $X, Y \in B(\mathbb{C}^n)$.

Writing $X$ in the basis $(\xi_0, \xi_1)$, we obtain

Writing $X$ in the basis $(\xi_0, \xi_1)$, we obtain

**Proposition**

$$\| T_{\phi}^{A_0,A_1} : \mathcal{S}_n^{\infty} \to \mathcal{S}_n^{\infty} \| = \left\| \{\phi(\lambda_i^{(0)}, \lambda_k^{(1)})\}_{i,k=1}^{n} : \mathcal{S}_n^{\infty} \to \mathcal{S}_n^{\infty} \right\|.$$

Writing $X$ in the basis $(\xi_0, \xi_1)$, we obtain

**Proposition**

$$\| T_\phi^{A_0,A_1} \colon \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty \| = \left\| \{\phi(\lambda_i^{(0)}, \lambda_k^{(1)})\}_{i,k=1}^n \colon \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty \right\|.$$

Writing $X$ in the basis $(\xi_0, \xi_1)$ and $Y$ in the basis $(\xi_1, \xi_2)$, we obtain the following result :

Writing $X$ in the basis $(\xi_0, \xi_1)$, we obtain

**Proposition**

$$\| T_{\phi}^{A_0, A_1} : \mathcal{S}_n^{\infty} \to \mathcal{S}_n^{\infty} \| = \left\| \{\phi(\lambda_i^{(0)}, \lambda_k^{(1)})\}_{i,k=1}^{n} : \mathcal{S}_n^{\infty} \to \mathcal{S}_n^{\infty} \right\|.$$

Writing $X$ in the basis $(\xi_0, \xi_1)$ and $Y$ in the basis $(\xi_1, \xi_2)$, we obtain the following result :

**Proposition**

$$\left\| T_{\psi}^{A_0, A_1, A_2} : \mathcal{S}_n^{2} \times \mathcal{S}_n^{2} \to \mathcal{S}_n^{1} \right\| = \left\| \{\psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)})\}_{i,j,k=1}^{n} : \mathcal{S}_n^{2} \times \mathcal{S}_n^{2} \to \mathcal{S}_n^{1} \right\|$$

• Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that $f$ admits right and left derivatives $f_r'(x)$ and $f_l'(x)$ at each $x \in \mathbb{R}$.
The divided difference of first order is defined by

$$f^{[1]}(x_0, x_1) := \begin{cases} \frac{f(x_0)-f(x_1)}{x_0-x_1}, & \text{if } x_0 \neq x_1 \\ \frac{f_r'(x_0)+f_l'(x_0)}{2} & \text{if } x_0 = x_1 \end{cases}, \quad x_0, x_1 \in \mathbb{R}.$$

• Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that $f$ admits right and left derivatives $f_r'(x)$ and $f_l'(x)$ at each $x \in \mathbb{R}$.
The divided difference of first order is defined by

$$f^{[1]}(x_0, x_1) := \begin{cases} \frac{f(x_0) - f(x_1)}{x_0 - x_1}, & \text{if } x_0 \neq x_1 \\ \frac{f_r'(x_0) + f_l'(x_0)}{2} & \text{if } x_0 = x_1 \end{cases}, \quad x_0, x_1 \in \mathbb{R}.$$

• If $f \in C^2(\mathbb{R})$, the divided difference of second order is defined by

$$f^{[2]}(x_0, x_1, x_2) := \begin{cases} \frac{f^{[1]}(x_0, x_1) - f^{[1]}(x_1, x_2)}{x_0 - x_2}, & \text{if } x_0 \neq x_2, \\ \frac{d}{dx_0} f^{[1]}(x_0, x_1), & \text{if } x_0 = x_2 \end{cases}, \quad x_0, x_1, x_2 \in \mathbb{R}.$$

## Proposition

*For all self-adjoint* $A_0, A_1 \in B(\mathbb{C}^n)$

$$f(A_0) - f(A_1) = T_{f^{[1]}}^{A_0,A_1}(A_0 - A_1).$$

## Proposition

*For all self-adjoint* $A_0, A_1 \in B(\mathbb{C}^n)$

$$f(A_0) - f(A_1) = T_{f^{[1]}}^{A_0,A_1}(A_0 - A_1).$$

*If* $f \in C^1(\mathbb{R})$*, the function* $t \mapsto f(A_0 + tA_1)$ *is differentiable and*

$$\frac{d}{dt}\big(f(A_0 + tA_1)\big)\Big|_{t=0} = T_{f^{[1]}}^{A_0,A_0}(A_1).$$

## Proposition

*For all self-adjoint* $A_0, A_1 \in B(\mathbb{C}^n)$

$$f(A_0) - f(A_1) = T_{f^{[1]}}^{A_0,A_1}(A_0 - A_1).$$

*If* $f \in C^1(\mathbb{R})$*, the function* $t \mapsto f(A_0 + tA_1)$ *is differentiable and*

$$\frac{d}{dt}\big(f(A_0 + tA_1)\big)\Big|_{t=0} = T_{f^{[1]}}^{A_0,A_0}(A_1).$$

## Theorem

*For all self-adjoints* $A, B \in B(\mathbb{C}^n)$ *and all function* $f \in C^2(\mathbb{R})$*, we have*

$$f(A+B) - f(A) - \frac{d}{dt}\big(f(A+tB)\big)\Big|_{t=0} = T_{f^{[2]}}^{A+B,A,A}(B, B).$$

Define $f_0(x) = |x|$.

**Theorem (Davies, 1988)**

*There exists a constant $C > 0$ such that for any $n \geq 1$, there exist $A_n, B_n \in \mathcal{B}(\mathbb{C}^{2n})$ self-adjoint such that $B_n \neq 0$ and*

$$\|f_0(A_n + B_n) - f_0(A_n)\|_1 \geq C \log n \|B_n\|_1.$$

Define $f_0(x) = |x|$.

**Theorem (Davies, 1988)**

*There exists a constant $C > 0$ such that for any $n \geq 1$, there exist $A_n, B_n \in \mathcal{B}(\mathbb{C}^{2n})$ self-adjoint such that $B_n \neq 0$ and*

$$\|f_0(A_n + B_n) - f_0(A_n)\|_1 \geq C \log n \|B_n\|_1.$$

Since $f_0(A_n + B_n) - f_0(A_n) = T_{f^{[1]}}^{A_n+B_n,A_n}(B_n)$ , this implies that

$$\left\| T_{f_0^{[1]}}^{A_n+B_n,A_n} : \mathcal{S}^1_{2n+1} \to \mathcal{S}^1_{2n+1} \right\| = \left\| T_{f_0^{[1]}}^{A_n+B_n,A_n} : \mathcal{S}^\infty_{2n+1} \to \mathcal{S}^\infty_{2n+1} \right\|$$
$$\geq C \ \log n.$$

Let $f \in C^2(\mathbb{R})$ be such that $f(x) = x|x|$ for $|x| \geq 1$ and $f^{(j)}(0) = 0$, $j = 0, 1, 2$.

Let $f \in C^2(\mathbb{R})$ be such that $f(x) = x|x|$ for $|x| \geq 1$ and $f^{(j)}(0) = 0$, $j = 0, 1, 2$.
By the equalities

$$\begin{aligned}
&\left\| T_{f^{[2]}}^{A+B,A,A} \colon \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1 \right\| \\
&= \left\| \{ f^{[2]}(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) \}_{i,j,k=1}^n \colon \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1 \right\| \\
&= \max_{1 \leq k \leq n} \left\| \{ f^{[1]}(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) \}_{i,k=1}^n \colon \ \colon M_n \to M_n \right\|. \\
&\geq C \log(n)
\end{aligned}$$

Let $f \in C^2(\mathbb{R})$ be such that $f(x) = x|x|$ for $|x| \geq 1$ and $f^{(j)}(0) = 0$, $j = 0, 1, 2$.
By the equalities

$$\begin{aligned}
&\left\| T_{f^{[2]}}^{A+B,A,A} \colon \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1 \right\| \\
&= \left\| \{ f^{[2]}(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) \}_{i,j,k=1}^n \colon \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1 \right\| \\
&= \max_{1 \leq k \leq n} \left\| \{ f^{[1]}(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) \}_{i,k=1}^n \colon \ \colon M_n \to M_n \right\|. \\
&\geq C \log(n)
\end{aligned}$$

and (lots of) technicalities, we obtain the estimate in the finite dimensional case.

Thanks for your attention !