S^1-BOUNDEDNESS OF TRIPLE OPERATOR INTEGRALS

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Joint work with Christian Le Merdy and Fedor Sukochev

Let H be a separable Hilbert space and let $T \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the set of compact operators on H. Then T^*T is compact, and so is $|T| = \sqrt{T^*T}$.
Moreover, $|T|$ is self-adjoint.

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There exists then a decreasing sequence of non-negative numbers $(\lambda_n(T))_n$ and an orthonormal family $(u_n)_n \in H$ such that

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By polar decomposition, there exists a partial isometry $U \in \mathcal{B}(H)$ such that $T = U|T|$.
Setting $v_n = U(u_n)$, we obtain an orthonormal family $(v_n)_n$ such that

$$T = \sum_n \lambda_n \langle ., u_n \rangle v_n.$$

Definition

We define, for $1 \leq p < \infty$,

$$S^p(H) = \{T \in \mathcal{K}(H) \mid (\lambda_n(T))_n \in \ell_p < \infty\}.$$

If $T \in S_p(H)$, we set $\|T\|_p = \|(\lambda_n(T))_n\|_{\ell_p}$. Notice that

$$\|T\|_=\mathrm{tr}(|T|^p)^{1/p}.$$

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Theorem

$(S_p(H), \|.\|_p)$ is a Banach space.

The elements of the space $S_2(H)$ are called Hilbert-Schmidt operators and the elements of $S_1(H)$ the trace class operators. We have a contraction $S_1(H) \subset S_2(H)$.

The case p=2 : let $T \in \mathcal{B}(\ell^2)$ and let $(t_{i,j})_{i,j\geq 1}$ be the associated infinite matrix. Then $T \in S_2(\ell^2)$ if and only if

$$\sum_{i,j\geq 1} |t_{i,j}|^2 < \infty.$$

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Let $(h_i)_{i\in\mathbb{N}}$ be a hilbertian basis of ℓ^2. Then $T \in S_2(\ell^2)$ if and only if

$$\sum_{i\in\mathbb{N}} \| T(h_i)\|^2 < \infty.$$

Let H be a separable Hilbert space.
Let A, B be normal operators on H (possibly unbounded).
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For A, this means that λ_A is a finite measure on $\sigma(A)$ such that if E_A denotes the spectral measure of A, then for any Borel subset $\Delta \subset \sigma(A)$,

$$\lambda_A(\Delta) = 0 \iff E_A(\Delta) = 0.$$

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Let

$$\Gamma^{A,B} : L^\infty(\lambda_A) \otimes L^\infty(\lambda_B) \longrightarrow \mathcal{B}(S_2(H), S_2(H))$$

defined by

$$\Gamma^{A,B}(f \otimes g)(X) = f(A)Xg(B)$$

for any $f \in L^\infty(\lambda_A), g \in L^\infty(\lambda_B)$ and $X \in S_2(H)$.

Then $\Gamma^{A,B}$ uniquely extends to a w^*-continuous isometry

$$\Gamma^{A,B} : L^\infty(\lambda_A \times \lambda_B) \longrightarrow \mathcal{B}(S_2(H), S_2(H)).$$

Operators of the form

$$\Gamma^{A,B}(\phi) : S_2(H) \longrightarrow S_2(H)$$

for $\phi \in L^\infty(\lambda_A \times \lambda_B)$ are called double operator integrals.

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The theory of double operator integrals started with Birman-Solomyak, in a series of three papers in 1966, 1967, 1973. Outstanding developments and applications were obtained by Peller, and by Sukochev and his co-authors in the last 20 years.

Let $M = (m_{ij})_{1\leq i,j\leq N}$ be a family of complex numbers. The associated **Schur multiplier** is the linear mapping $T_M : M_N \to M_N$ defined by

$$T_M([a_{ij}]) = [m_{ij}a_{ij}]_{1\leq i,j\leq N}, A = [a_{i,j}]_{1\leq i,j\leq N} \in M_N.$$

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Theorem

We have $\|T_M\| < 1$ if and only if there exist a Hilbert space K and two families $\{a_j\}_{1\le j\le N}, \{b_i\}_{1\le i\le N}$ of elements K such that

$$m_{ij} = \langle b_i, a_j\rangle, i,j = 1,\dots,N$$

with

$$\|a_j\| < 1, \|b_i\| < 1, i,j = 1,\dots,N.$$

Assume that H is finite dimensional, say of dimension N. Let

$$A = \sum_{i=1}^{N} \lambda_i P_i \quad \text{and} \quad B = \sum_{j=1}^{N} \mu_j Q_j$$

be the spectral decompositions of A and B.
Meaning for A : $(e_1, \ldots, e_N)$ is a orthonormal basis of eigenvectors of A, P_i are the corresponding orthogonal projections onto $\mathbb{C}e_i$ and λ_i are the corresponding eigenvalues.

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Then, for any $\phi : \mathbb{C}^2 \to \mathbb{C}$,

$$[\Gamma^{A,B}(\phi)](X) = \sum_{i,j=1}^{N} \phi(\lambda_i, \mu_j) P_i X Q_j, X \in M_N.$$

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Hence, $\Gamma^{A,B}(\phi)$ behaves like the Schur multiplier associated with the family

$$(\phi(\lambda_i, \mu_j))_{1 \leq i,j \leq N}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$-function such that f' is bounded.
Let $f^{[1]} : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f^{[1]}(x,y) := \begin{cases} \frac{f(x)-f(y)}{x-y}, & \text{if } x \neq y \\ f'(x), & \text{if } x = y \end{cases}, \quad x_0, x_1 \in \mathbb{R}.$$

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Then $f^{[1]}$ is a bounded continuous function.
Let A, D be selfadjoint operators with $D \in S_2(H)$. Then $f^{[1]} \in L^\infty(\lambda_{A+D} \times \lambda_A)$ and

$$[\Gamma^{A+D,A}(f^{[1]})](D) = f(A+D) - f(A).$$

It is as if we had

$$\Gamma^{A+D,A}(f^{[1]}) = \frac{f(A+D) - f(A)}{(A+D) - A}.$$

Let A, B be self-adjoint on H.
If f is Lipschitz on $\mathbb{R}$ and $B \in S^1(H)$, do we have

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Counter-example by Farforovskaya (1972) : there exist $f \in \mathcal{C}^1(\mathbb{R})$, A, B bounded self-adjoints operators with $B \in S^1(H)$ such that

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Theorem (D. Potapov, F. Sukochev, 2011)

Let $1 < p < \infty$. There exists a constant $c_p > 0$ such that for any Lipschitz function $f : \mathbb{R} \to \mathbb{C}$ and for all A, B self-adjoints with $B \in S_p(H)$,

$$\|f(A+B) - f(A)\|_p \leq c_p \|f\|_{Lip_1} \|B\|_p.$$

Let A, B be normal operators on a separable Hilbert space H.
Let $S_1(H)$ be the space of trace class operators on H.

Theorem (Peller's Theorem)

For any $\phi \in L^\infty(\lambda_A \times \lambda_B)$, the following are equivalent :
(i) $\Gamma^{A,B}(\phi)$ restricts to a bounded map $S_1(H) \to S_1(H)$.
(ii) There exist a Hilbert space K and two functions $a \in L^\infty(\lambda_A; K)$ and $b \in L^\infty(\lambda_B; K)$ such that, for a.e. (s,t),

$$\phi(s,t) = \langle a(s), b(t) \rangle .$$

In this case, $\|\Gamma^{A,B}(\phi) : S_1(H) \to S_1(H)\| = \inf \|a\| \|b\|$.

Recall the identification

$$L^\infty(\lambda_A \times \lambda_B) \simeq \mathcal{B}(L^1(\lambda_A), L^\infty(\lambda_B)).$$

To any $\phi \in L^\infty(\lambda_A \times \lambda_B)$, one associates $u_\phi : L^1(\lambda_A) \to L^\infty(\lambda_B)$ by

$$u_\phi(f) = \int_{\sigma(A)} \phi(s, \cdot) f(s) \, \mathrm{d}\lambda_A(s).$$

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Then (ii) means that u_ϕ factors through a Hilbert space. Indeed, associate $\alpha : L^1(\lambda_A) \to K$ to $a \in L^\infty(\lambda_A; K)$ by

$$\alpha(f) = \int_{\sigma(A)} a(s) f(s)\, \mathrm{d}\lambda_A(s).$$

Similarly, associate $\beta : K \to L^\infty(\lambda_B)$ to $b \in L^\infty(\lambda_B; K)$.

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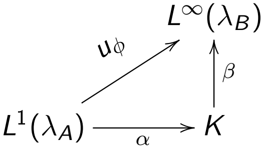
Then, the identity

$$\phi(s,t) = \langle a(s), b(t) \rangle$$

says that

$$u_\phi = \beta\alpha.$$



Let H be a separable Hilbert space.
Let A, B, C be three normal operators on H.
Let $\lambda_A, \lambda_B, \lambda_C$ be scalar valued spectral measures for A, B, C.
Write $S_2 = S_2(H)$ and let $\mathcal{B}_2(S_2 \times S_2, S_2)$ be the space of bilinear maps from $S_2 \times S_2$ into S_2.

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Let

$$\Gamma^{A,B,C} : L^\infty(\lambda_A) \otimes L^\infty(\lambda_B) \otimes L^\infty(\lambda_C) \longrightarrow \mathcal{B}_2(S_2 \times S_2, S_2)$$

defined by

$$\Gamma^{A,B,C}(f \otimes g \otimes h)(X) = f(A)Xg(B)Yh(C).$$

for any $f \in L^\infty(\lambda_A), g \in L^\infty(\lambda_B), h \in L^\infty(\lambda_C)$ and $X, Y \in S_2(H)$.

Theorem

$\Gamma^{A,B,C}$ uniquely extends to a w^*-continuous isometry

$$\Gamma^{A,B,C} : L^\infty(\lambda_A \times \lambda_B \times \lambda_C) \longrightarrow \mathcal{B}_2(S_2 \times S_2, S_2).$$

Such constructions go back (at least) to Pavlov (1969).

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Such constructions go back (at least) to Pavlov (1969).
Operators of the form

$$\Gamma^{A,B,C}(\phi) : S_2 \times S_2 \longrightarrow S_2$$

for $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$ are called **triple operator integrals**.

Alternative definitions of multiple operator integrals appear in important papers by Peller and also by Potapov-Skripka-Sukochev.

Let $N \geq 1$ be an integer.
Let $M = \{m_{ikj}\}_{1 \leq i,k,j \leq N}$ be a family of complex numbers.
The associated bilinear Schur multipliers

$$T_M : M_N \times M_N \to M_N$$

is the bilinear map defined by

$$T_M([x_{ij}], [y_{ij}]) = \left[\sum_k m_{ikj} a_{ik} b_{kj} \right]_{1 \leq i,j \leq N}.$$

Let A, B, C be normal operators on H, and set $N = \dim(H)$. Let

$$A = \sum_{i=1}^{N} \lambda_i P_i \quad \text{and}, B = \sum_{k=1}^{N} \mu_k Q_k \quad \text{and} \quad C = \sum_{j=1}^{N} \nu_j R_j$$

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Proposition

For any $\phi : \mathbb{C}^3 \to \mathbb{C}$ and any matrices X and Y,

$$[\Gamma^{A,B,C}(\phi)](X, Y) = \sum_{i,j,k=1}^{N} \phi(\lambda_i, \mu_k, \nu_j) P_i X Q_k Y R_j, X \in M_N.$$

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This shows that $\Gamma^{A,B,C}(\phi)$ behaves like the bilinear Schur multiplier associated with the family

$$(\phi(\lambda_i, \mu_k, \nu_j))_{1 \le i,k,j \le N}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$-function such that f'' is bounded.
Let $f^{[2]} : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f^{[2]}(x, y, z) := \begin{cases} \frac{f^{[1]}(x,y) - f^{[1]}(y,z)}{x-z}, & \text{if } x \neq z \\ \dfrac{\partial}{\partial x} f^{[1]}(x, y), & \text{if } x = z \end{cases}.$$

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Let A, D be selfadjoint operators with $D \in S_2(H)$. Assume that A is bounded, so that we can define separately

$$f(A + D) - f(A) \ \text{ and } \ \frac{d}{dt}(f(A + tD))_{|t=0}.$$

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PROPOSITION

$$f(A+D)-f(A)-\frac{d}{dt}(f(A+tD))_{|t=0} = [\Gamma^{A+D,A,A}(f^{[2]})](D,D).$$

What are the functions ϕ for which $\Gamma^{A,B,C}(\phi)$ maps $S_2 \times S_2$ into S_1 ?

Theorem (C., Le Merdy, Potapov, Sukochev, Tomskova)

Let $M = \{m_{ikj}\}_{1 \leq i,k,j \leq N}$ be a family of complex numbers. Consider the bilinear Schur multiplication

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Then $\|T_M\| < 1$ if and only if there exist a Hilbert space K and two families $\{a_{ik}\}_{1 \leq i,k \leq N}$ and $\{b_{jk}\}_{1 \leq j,k \leq N}$ of elements of K such that

$$m_{ikj} = \langle a_{ik}, b_{jk} \rangle , \quad i,k,j \geq 1$$

and

$$\|a_{ik}\| < 1 \quad \text{and} \quad \|b_{jk}\| < 1, \quad i,k,j \geq 1.$$

Let A, B, C be normal operators on a separable Hilbert space H.

Main Theorem

For any $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$, the following are equivalent :
(i) $\Gamma^{A,B,C}(\phi)$ is a bounded bilinear map $S_2(H) \times S_2(H) \to S_1(H)$.

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For any $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$*, the following are equivalent :*
(i) $\Gamma^{A,B,C}(\phi)$ *is a bounded bilinear map* $S_2(H) \times S_2(H) \to S_1(H)$*.*
(ii) There exist a Hilbert space K *and two functions*

$$a \in L^\infty(\lambda_A \times \lambda_B; K) \quad \text{and} \quad b \in L^\infty(\lambda_B \times \lambda_C; K)$$

such that

$$\phi(r,s,t) = \langle a(r,s), b(s,t) \rangle \quad \text{for a.e. } (r,s,t).$$

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In this case,

$$\|\Gamma^{A,B,C}(\phi) : S_2(H) \times S_2(H) \to S_1(H)\| = \inf \|a\| \|b\|.$$

• Given Banach spaces E, F , we let $\Gamma_2(E, F)$ to be the space of operators $T : E \to F$ which factor through a Hilbert space, i.e. there exist a Hilbert space K and two operators $\alpha : E \to K$ and $\beta : K \to F$ such that $T = \beta\alpha$.
This is a Banach space for the norm

$$\gamma_2(T) = \inf \|\alpha\| \|\beta\|.$$

When F is a dual space, $\Gamma_2(E, F)$ is also a dual space.

• Given Banach spaces E, F , we let $\Gamma_2(E, F)$ to be the space of operators $T : E \to F$ which factor through a Hilbert space, i.e. there exist a Hilbert space K and two operators $\alpha : E \to K$ and $\beta : K \to F$ such that $T = \beta\alpha$.
This is a Banach space for the norm

$$\gamma_2(T) = \inf \|\alpha\|\|\beta\|.$$

When F is a dual space, $\Gamma_2(E, F)$ is also a dual space.

• Let (Σ, λ) be a σ-finite measure space and let G be a separable Banach space. We let $L^\infty_\sigma(\lambda; G^*)$ be the space of all essentially bounded w^*-measurable functions $\psi : \Sigma \to G^*$, and set

$$\|\psi\| = \text{ess sup}_{s \in S} \|\psi(s)\|$$

for such fonctions. Then after taking quotient by almost everywhere zero functions this is a Banach space and we have

$$L^\infty_\sigma(\lambda, G^*) = L^1(\lambda; G)^*.$$

Theorem (Factorization Theorem)

Let (Σ, λ) be a σ-finite measure space, let E, F be separable Banach spaces. Let $\psi \in L^\infty_\sigma(\lambda, \Gamma_2(E, F^*))$. Then there exist a Hilbert space K, and two functions

$$a \in L^\infty_\sigma(\lambda; \mathcal{B}(E, K)) \quad \text{and} \quad b \in L^\infty_\sigma(\lambda; \mathcal{B}(F, K))$$

such that $\|a\|\|b\| \leq \|\psi\|$ and for any $e \in E$ and $f \in F$,

$$\langle [\psi(s)](e), f \rangle = \langle [a(s)](e), [b(s)](f) \rangle$$

almost everywhere.

Regard $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$ as an element of $L^\infty_\sigma(\lambda_B; L^\infty(\lambda_A \times \lambda_C)$ and associate

$$\psi \in L^\infty_\sigma(\lambda_B; \mathcal{B}(L^1(\lambda_A); L^\infty(\lambda_C)))$$

by

$$[\psi(s)](f) = \int_{\sigma(A)} \phi(r, s, \cdot) f(r) \, \mathrm{d}\lambda_A(r)$$

for any $f \in L^1(\lambda_A)$.

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Main Theorem (Second formulation)

The following are equivalent :
(i) $\Gamma^{A,B,C}(\phi)$ is a bounded bilinear map $S_2(H) \times S_2(H) \to S_1(H)$.
(ii) The function ψ associated to ϕ belongs to

$$L^\infty_\sigma(\lambda_B; \Gamma_2(L^1(\lambda_A); L^\infty(\lambda_C))).$$

Thank you for your attention !