

PERTURBATION OF SECOND ORDER FOR FUNCTIONAL CALCULUS

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THEOREM (D. POTAPOV, F. SUKOCHEV, 2011)

Let $1 < p < \infty$. There exists a constant $c_p > 0$ such that for any Lipschitz function $f : \mathbb{R} \rightarrow \mathbb{C}$ and for all A, B self-adjoints with $B \in S_p(H)$,

$$\|f(A + B) - f(A)\|_p \leq c_p \|f\|_{Lip_1} \|B\|_p.$$

Let A, B be self-adjoint operators and $f \in \mathcal{C}^2(\mathbb{R})$ such that $\|f''\|_\infty < \infty$.

Write

$$\Gamma(A, B, f) = f(A + B) - f(A) - \frac{d}{dt}(f(A + tB))\Big|_{t=0}.$$

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If A is bounded and $B \in S^2(\mathcal{H})$, then

$$f(A + B) - f(A) \text{ et } \left. \frac{d}{dt}(f(A + tB)) \right|_{t=0}$$

are well defined and belong to $S^2(\mathcal{H})$.

If A is unbounded. Take $f : x \mapsto x^2$. We have

$$f(A + B) - f(A) = AB + BA + B^2 \quad \text{and} \quad \frac{d}{dt}(f(A + tB))\Big|_{t=0} = AB + BA.$$

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THEOREM (V. PELLER, 2004)

There exists $C > 0$ such that for all $f \in B_{\infty 1}^2$,

$$\|\Gamma(A, B, f)\|_1 \leq C \|f\|_{B_{\infty 1}^2} \|B\|_2^2.$$

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THEOREM (C., LE MERDY, POTAPOV, SUKOCHEV, TOMSKOVA, 2015)

There exist an unbounded self-adjoint operator A and a self-adjoint operator $B \in \mathcal{S}^2(\ell^2)$ such that

$$\Gamma(A, B, f) = f(A + B) - f(A) - \frac{d}{dt}(f(A + tB))\Big|_{t=0} \notin \mathcal{S}^1.$$

A and B will be of the form $A = \bigoplus_{n \geq 1} A_n$, $B = \bigoplus_{n \geq 1} B_n$ where $A_n, B_n \in \mathcal{B}(\mathcal{H}_n)$ with \mathcal{H}_n finite dimensional.

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We will then have

$$\Gamma(A, B, f) = \bigoplus_{n \geq 1} \left(f(A_n + B_n) - f(A_n) - \frac{d}{dt}(f(A_n + tB_n)) \Big|_{t=0} \right).$$

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LEMMA

There exist $C > 0$ and two sequences of operators $A_n, B_n \in B(\mathbb{C}^{8n+4})$ such that $\|B_n\|_2^2 \leq \frac{1}{n \log^{3/2} n}$, for all $n \geq N$, and

$$\|f(A_n + B_n) - f(A_n) - \frac{d}{dt}(f(A_n + tB_n)) \Big|_{t=0}\|_1 \geq \frac{C}{n \log^{1/2}(n)}.$$

Soit $1 \leq p \leq \infty$.

- A matrix $N = \{n_{ij}\}_{i,j \geq 1}$ with entries in \mathbb{C} is called a Schur multiplier on S^p if the following action

$$S_N(A) := \sum_{i,j \geq 1} n_{ij} a_{ij} E_{ij}, \quad A = \{a_{ij}\}_{i,j \geq 1} \in S^p,$$

defines a bounded operator on $S^p(\ell^2)$.

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- A three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j \geq 1}$ with entries in \mathbb{C} is called a bilinear Schur multiplier onto S^p if the following action

$$T_M(A, B) := \sum_{i,j,k \geq 1} m_{ikj} a_{ik} b_{kj} E_{ij}, \quad A = \{a_{ij}\}_{i,j \geq 1}, B = \{b_{ij}\}_{i,j \geq 1} \in \mathcal{S}^2,$$

defines a bounded bilinear mapping from $S^2 \times S^2$ onto $S^p(\ell^2)$.

Remarks : • There is a description of linear Schur multipliers on $\mathcal{B}(\ell^2)$ (Grothendieck's theorem).

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• If $N = \{n_{ij}\}_{i,j \geq 1}$ is a linear Schur multipliers on \mathcal{S}^p then $\sup_{i,j \geq 1} |n_{ij}| < \infty$. For $p = 2$, this condition is sufficient and we have,

$$\|S_N : \mathcal{S}^2 \rightarrow \mathcal{S}^2\| = \sup_{i,j \geq 1} |n_{ij}|.$$

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PROPOSITION

Let $n \in \mathbb{N}$. Let $M = \{m_{ikj}\}_{i,k,j=1}^n$ be a three-dimensional matrix and for all $1 \leq k \leq n$, denote $T(k)$ the matrix given by $T(k) = \{m_{ikj}\}_{i,j=1}^n$. Then

$$\|T_M : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \rightarrow \mathcal{S}_n^1\| = \max_{1 \leq k \leq n} \|S_{T(k)} : M_n \rightarrow M_n\|.$$

Let $A_0, A_1 \in B(\mathbb{C}^n)$ be diagonalizable self-adjoint operators. For $j = 0, 1$, let $\xi_j = \{\xi_i^{(j)}\}_{i=1}^n$ be an orthonormal basis of eigenvectors for A_j , and let $\{\lambda_i^{(j)}\}_{i=1}^n$ be the associated n -tuple of eigenvalues.

Denote $P_{\xi_i^{(j)}} = \xi_i^{(j)} \otimes \overline{\xi_i^{(j)}}$ the projection onto $\mathbb{C}\xi_i^{(j)}$.

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Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{C}$ be a bounded Borel function. Define a linear operator $T_\phi^{A_0, A_1} : B(\mathbb{C}^n) \rightarrow B(\mathbb{C}^n)$ given by

$$T_\phi^{A_0, A_1}(X) = \sum_{i,k=1}^n \phi(\lambda_i^{(0)}, \lambda_k^{(1)}) P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}}, \quad X \in B(\mathbb{C}^n).$$

Let $A_0, A_1, A_2 \in B(\mathbb{C}^n)$ be diagonalizable self-adjoint operators and for any $j = 0, 1, 2$, let $\xi_j = \{\xi_i^{(j)}\}_{i=1}^n$ be an orthonormal basis of eigenvectors of A_j and let $\{\lambda_i^{(j)}\}_{i=1}^n$ be the corresponding n -tuple of eigenvalues.

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Let $\psi : \mathbb{R}^3 \rightarrow \mathbb{C}$ be a bounded Borel function. We define the bilinear mapping

$$T_{\psi}^{A_0, A_1, A_2}(X, Y) = \sum_{i, j, k=1}^n \psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}} Y P_{\xi_j^{(2)}}$$

for any $X, Y \in B(\mathbb{C}^n)$.

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PROPOSITION

$$\|T_{\psi}^{A_0, A_1, A_2} : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \rightarrow \mathcal{S}_n^1\| = \|\{\psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)})\}_{i,j,k=1}^n : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \rightarrow \mathcal{S}_n^1\|$$

- Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and assume that f admits right and left derivatives $f'_r(x)$ and $f'_l(x)$ at each $x \in \mathbb{R}$.

The divided difference of first order is defined by

$$f^{[1]}(x_0, x_1) := \begin{cases} \frac{f(x_0) - f(x_1)}{x_0 - x_1}, & \text{if } x_0 \neq x_1 \\ \frac{f'_r(x_0) + f'_l(x_0)}{2} & \text{if } x_0 = x_1 \end{cases}, \quad x_0, x_1 \in \mathbb{R}.$$

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- If $f \in C^2(\mathbb{R})$, the divided difference of second order is defined by

$$f^{[2]}(x_0, x_1, x_2) := \begin{cases} \frac{f^{[1]}(x_0, x_1) - f^{[1]}(x_1, x_2)}{x_0 - x_2}, & \text{if } x_0 \neq x_2 \\ \frac{d}{dx_0} f^{[1]}(x_0, x_1), & \text{if } x_0 = x_2 \end{cases}, \quad x_0, x_1, x_2 \in \mathbb{R}.$$

PROPOSITION

For all self-adjoint $A_0, A_1 \in B(\mathbb{C}^n)$

$$f(A_0) - f(A_1) = T_{f[1]}^{A_0, A_1}(A_0 - A_1).$$

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If $f \in C^1(\mathbb{R})$, the function $t \mapsto f(A_0 + tA_1)$ is differentiable and

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THEOREM

For all self-adjoints $A, B \in B(\mathbb{C}^n)$ and all function $f \in C^2(\mathbb{R})$, we have

$$f(A + B) - f(A) - \left. \frac{d}{dt}(f(A + tB)) \right|_{t=0} = T_{f[2]}^{A+B, A, A}(B, B).$$

Define $f_0(x) = |x|$.

THEOREM (DAVIES, 1988)

There exists a constant $C > 0$ such that for any $n \geq 1$, there exist $A_n, B_n \in \mathcal{B}(\mathbb{C}^{2^n})$ self-adjoint such that $B_n \neq 0$ and

$$\|f_0(A_n + B_n) - f_0(A_n)\|_1 \geq C \log n \|B_n\|_1.$$

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Since $f_0(A_n + B_n) - f_0(A_n) = T_{f_0^{[1]}}^{A_n+B_n, A_n}(B_n)$, this implies that

$$\begin{aligned} \left\| T_{f_0^{[1]}}^{A_n+B_n, A_n} : \mathcal{S}_{2n+1}^1 \rightarrow \mathcal{S}_{2n+1}^1 \right\| &= \left\| T_{f_0^{[1]}}^{A_n+B_n, A_n} : \mathcal{S}_{2n+1}^\infty \rightarrow \mathcal{S}_{2n+1}^\infty \right\| \\ &\geq C \log n. \end{aligned}$$

Let $f \in C^2(\mathbb{R})$ be such that $f(x) = x|x|$ for $|x| \geq 1$ and $f^{(j)}(0) = 0$, $j = 0, 1, 2$.

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By the equalities

$$\begin{aligned}
 & \left\| T_{f^{[2]}}^{A+B, A, A} : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \rightarrow \mathcal{S}_n^1 \right\| \\
 &= \left\| \{f^{[2]}(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)})\}_{i,j,k=1}^n : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \rightarrow \mathcal{S}_n^1 \right\| \\
 &= \max_{1 \leq k \leq n} \left\| \{f^{[1]}(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)})\}_{i,j,k=1}^n : M_n \rightarrow M_n \right\|. \\
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and (lots of) technicalities, we obtain the estimate in the finite dimensional case.

Thanks for your attention !