Triple operator integrals valued in trace class operators

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Joint work with Christian Le Merdy and Fedor Sukochev

- Let $1 \leq p < +\infty$.

$$\ell_p = \left\{ x = (x_n)_{n=1}^{+\infty} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

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- If $1 \leq p \leq q \leq \infty$,

$$\ell_1 \subset \ell_p \subset \ell_q \subset \ell_\infty.$$

Let X and Y be normed spaces.

- A linear operator $T : X \to Y$ is called **bounded** if there exists a constant $C \geq 0$ such that for all $x \in X$,

$$\|T(x)\|_Y \leq C\|x\|_X.$$

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$\|T\| = \inf C$.
- We denote by $\mathcal{B}(X, Y)$ the space of all bounded linear operators from X into Y.
If $X = Y$, we simply write $\mathcal{B}(X)$.

- For every operator $Q \in \mathcal{B}(\ell_p, \ell_q)$ there exists a (infinite) matrix $A = (a_{jk})_{j,k=1}^{\infty}$ such that $Q(\{x_n\}_{n=1}^{\infty}) = A\{x_n\}_{n=1}^{\infty}$.

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- $$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{\infty} a_{1k} x_k \\ \sum_{k=1}^{\infty} a_{2k} x_k \\ \sum_{k=1}^{\infty} a_{3k} x_k \\ \vdots \end{pmatrix}$$

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- From now on, we identify any element of $\mathcal{B}(\ell_p, \ell_q)$ with its corresponding matrix.

Let H be a separable Hilbert space and let $T \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the set of compact operators on H. Then T^*T is compact, and so is $|T| = \sqrt{T^*T}$.
Moreover, $|T|$ is self-adjoint.

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There exists then a decreasing sequence of non-negative numbers $(\lambda_n(T))_n$ and an orthonormal family $(u_n)_n \in H$ such that

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By polar decomposition, there exists a partial isometry $U \in \mathcal{B}(H)$ such that $T = U|T|$.
Setting $v_n = U(u_n)$, we obtain an orthonormal family $(v_n)_n$ such that

$$T = \sum_n \lambda_n \langle ., u_n \rangle v_n.$$

DEFINITION

We define, for $1 \leq p < \infty$,

$$S^p(H) = \{T \in \mathcal{K}(H) \mid (\lambda_n(T))_n \in \ell_p\}.$$

If $T \in S^p(H)$, we set $\|T\|_p = \|(\lambda_n(T))_n\|_{\ell_p}$.

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Let $1 \leq p \leq q < \infty$. We have

$$S^1(\mathcal{H}) \subset S^p(\mathcal{H}) \subset S^q(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}).$$

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$$S^1(\mathcal{H}) \subset S^p(\mathcal{H}) \subset S^q(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}).$$

The elements of the space $S^2(H)$ are called Hilbert-Schmidt operators and the elements of $S^1(H)$ the trace class operators.

Example : Let $T = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_n & \\ & & & \ddots \end{pmatrix}$ be the finite or infinite diagonal matrix with diagonal $(\lambda_n)_n$.

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Then

$$T \in S^p \quad \text{if and only if} \quad (\lambda_n)_n \in \ell_p$$

and $\|T\|_p = \|(\lambda_n)_n\|_p$.

The case p=2 : let $T \in \mathcal{B}(\ell^2)$ and let $(t_{i,j})_{i,j\geq 1}$ be the associated infinite matrix. Then $T \in S^2(\ell^2)$ if and only if

$$\sum_{i,j\geq 1} |t_{i,j}|^2 < \infty.$$

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Let $(h_i)_{i\in\mathbb{N}}$ be a hilbertian basis of ℓ_2. Then $T \in S^2(\ell_2)$ if and only if

$$\sum_{i\in\mathbb{N}} \|T(h_i)\|^2 < \infty.$$

- Let $M = (m_{ij})_{1\leq i,j}$ be a family of complex numbers.
  We say that M is a Schur multiplier on S^p (resp. on $\mathcal{B}(\ell_p, \ell_q)$) if for any matrix $[a_{ij}] \in S^p$ (resp. in $\mathcal{B}(\ell_p, \ell_q)$), the Schur product of M and A

$$M * A = [m_{ij} a_{ij}]$$

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- If M is a Schur multiplier then M is bounded.
- If $p = 2$, then this condition is sufficient on S^2.

Example : The main triangular truncation. Let

$$M = \begin{pmatrix} 1 & 1 & \dots & & \\ 0 & 1 & 1 & \dots & \\ 0 & 0 & 1 & 1 & \dots \\ \dots & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

If $A = (a_{ik})$, then $M * A = \begin{pmatrix} a_{11} & a_{12} & \dots & & \\ 0 & a_{22} & a_{23} & \dots & \\ 0 & 0 & a_{33} & a_{34} & \dots \\ \dots & 0 & 0 & a_{44} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$

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- In 1970, Kwapień and Pelczyński proved that if $q \leq p$, M is not a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$.
- In 1976, Bennett proved that if $p < q$, M is a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$.

There is a famous characterization of Schur multipliers on $\mathcal{B}(\ell_2)$.

Theorem

Let $M = (m_{ij})_{i,j\in\mathbb{N}} \subset \mathbb{C}$. The following are equivalent :
(i) M is a Schur multiplier on $\mathcal{B}(\ell_2)$.
(ii) There is a Hilbert space $\mathcal{H}$ and two bounded sequences $(x_j)_{j\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ of elements of $\mathcal{H}$ such that

$$\forall i,j \in \mathbb{N},\ \ m_{ij} = \langle x_j, y_i \rangle \, .$$

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For A, this means that λ_A is a finite measure on $\sigma(A)$ such that if E_A denotes the spectral measure of A, then for any Borel subset $\Delta \subset \sigma(A)$,

$$\lambda_A(\Delta) = 0 \Longleftrightarrow E_A(\Delta) = 0.$$

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Let

$$\Gamma^{A,B} : L^\infty(\lambda_A) \otimes L^\infty(\lambda_B) \longrightarrow \mathcal{B}(S^2(H), S^2(H))$$

defined by

$$\Gamma^{A,B}(f \otimes g)(X) = f(A)Xg(B)$$

for any $f \in L^\infty(\lambda_A), g \in L^\infty(\lambda_B)$ and $X \in S^2(H)$.

Then $\Gamma^{A,B}$ uniquely extends to a w^*-continuous isometry

$$\Gamma^{A,B} : L^\infty(\lambda_A \times \lambda_B) \longrightarrow \mathcal{B}(S^2(H), S^2(H)).$$

Operators of the form

$$\Gamma^{A,B}(\phi) : S^2(H) \longrightarrow S^2(H)$$

for $\phi \in L^\infty(\lambda_A \times \lambda_B)$ are called double operator integrals.

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The theory of double operator integrals started with Birman-Solomyak, in a series of three papers in 1966, 1967, 1973. Outstanding developments and applications were obtained by Peller, and by Sukochev and his co-authors in the last 20 years.

Assume that H is finite dimensional, say of dimension N. Let

$$A = \sum_{i=1}^{N} \lambda_i P_i \quad \text{and} \quad B = \sum_{j=1}^{N} \mu_j Q_j$$

be the spectral decompositions of A and B.
Meaning for $A : (e_1, \ldots, e_N)$ is a orthonormal basis of eigenvectors of A, P_i are the corresponding orthogonal projections onto $\mathbb{C}e_i$ and λ_i are the corresponding eigenvalues.

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Then, for any $\phi : \mathbb{C}^2 \to \mathbb{C}$,

$$[\Gamma^{A,B}(\phi)](X) = \sum_{i,j=1}^{N} \phi(\lambda_i, \mu_j) P_i X Q_j, X \in M_N.$$

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Hence, $\Gamma^{A,B}(\phi)$ behaves like the Schur multiplier associated with the family

$$(\phi(\lambda_i, \mu_j))_{1 \leq i,j \leq N}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$-function such that f' is bounded.
Let $f^{[1]} : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f^{[1]}(x,y) := \begin{cases} \frac{f(x)-f(y)}{x-y}, & \text{if } x \neq y \\ f'(x), & \text{if } x = y \end{cases}, \quad x_0, x_1 \in \mathbb{R}.$$

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Then $f^{[1]}$ is a bounded continuous function.
Let A, D be selfadjoint operators with $D\in S^2(H)$. Then

$$[\Gamma^{A+D,A}(f^{[1]})](D) = f(A+D) - f(A).$$

It is as if we had

$$\Gamma^{A+D,A}(f^{[1]}) = \frac{f(A+D)-f(A)}{(A+D)-A}.$$

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If f is Lipschitz on $\mathbb{R}$ and $B \in S^p(H)$, do we have

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Theorem (D. Potapov, F. Sukochev, 2011)

Let $1 < p < \infty$. There exists a constant $c_p > 0$ such that for any Lipschitz function $f : \mathbb{R} \to \mathbb{C}$ and for all A, B self-adjoints with $B \in S_p(H)$,

$$\|f(A+B) - f(A)\|_p \leq c_p \|f\|_{Lip_1} \|B\|_p.$$

Let A, B be normal operators on a separable Hilbert space H.
Let $S^1(H)$ be the space of trace class operators on H.

Theorem (Peller's Theorem)

For any $\phi \in L^\infty(\lambda_A \times \lambda_B)$, the following are equivalent :
(i) $\Gamma^{A,B}(\phi)$ restricts to a bounded map $S^1(H) \to S^1(H)$.
(ii) There exist a Hilbert space K and two functions $a \in L^\infty(\lambda_A; K)$ and $b \in L^\infty(\lambda_B; K)$ such that, for a.e. (s,t),

$$\phi(s,t) = \langle a(s), b(t) \rangle .$$

In this case, $\|\Gamma^{A,B}(\phi) : S^1(H) \to S^1(H)\| = \inf \|a\| \|b\|$.

- A three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ with entries in $\mathbb{C}$ is called a bilinear Schur multiplier onto S^p if the following action
$$T_M(A,B) := \sum_{i,j,k\geq 1} m_{ikj} a_{ik} b_{kj} E_{ij},$$
$A = \{a_{ij}\}_{i,j\geq 1}, B = \{b_{ij}\}_{i,j\geq 1} \in \mathcal{S}^2$,, defines a bounded bilinear mapping from $S^2 \times S^2$ onto $S^p(\ell^2)$.

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- If $p = 2$, then this condition is sufficient on S^2.

Let H be a separable Hilbert space.
Let A, B, C be three normal operators on H.
Let $\lambda_A, \lambda_B, \lambda_C$ be scalar valued spectral measures for A, B, C.
Write $S^2 = S^2(H)$ and let $\mathcal{B}_2(S^2 \times S^2, S^2)$ be the space of bilinear maps from $S^2 \times S^2$ into S^2.

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Let

$$\Gamma^{A,B,C} : L^\infty(\lambda_A) \otimes L^\infty(\lambda_B) \otimes L^\infty(\lambda_C) \longrightarrow \mathcal{B}_2(S^2 \times S^2, S^2)$$

defined by

$$\left[\Gamma^{A,B,C}(f \otimes g \otimes h)\right](X, Y) = f(A)Xg(B)Yh(C).$$

for any $f \in L^\infty(\lambda_A), g \in L^\infty(\lambda_B), h \in L^\infty(\lambda_C)$ and $X, Y \in S^2(H)$.

Theorem

$\Gamma^{A,B,C}$ *uniquely extends to a* w^*-*continuous isometry*

$$\Gamma^{A,B,C} : L^\infty(\lambda_A \times \lambda_B \times \lambda_C) \longrightarrow \mathcal{B}_2(S^2 \times S^2, S^2).$$

Such constructions go back (at least) to Pavlov (1969).

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Such constructions go back (at least) to Pavlov (1969).
Operators of the form

$$\Gamma^{A,B,C}(\phi) : S^2 \times S^2 \longrightarrow S^2$$

for $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$ are called **triple operator integrals**.

Alternative definitions of multiple operator integrals appear in important papers by Peller and also by Potapov-Skripka-Sukochev.

Let A, B, C be normal operators on H, and set $N = \dim(H)$. Let

$$A = \sum_{i=1}^{N} \lambda_i P_i \quad \text{and}, B = \sum_{k=1}^{N} \mu_k Q_k \quad \text{and} \quad C = \sum_{j=1}^{N} \nu_j R_j$$

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Proposition

For any $\phi : \mathbb{C}^3 \to \mathbb{C}$ and any matrices X and Y,

$$[\Gamma^{A,B,C}(\phi)](X, Y) = \sum_{i,j,k=1}^{N} \phi(\lambda_i, \mu_k, \nu_j) P_i X Q_k Y R_j, X \in M_N.$$

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This shows that $\Gamma^{A,B,C}(\phi)$ behaves like the bilinear Schur multiplier associated with the family

$$(\phi(\lambda_i, \mu_k, \nu_j))_{1 \leq i,k,j \leq N}.$$

What are the functions ϕ for which $\Gamma^{A,B,C}(\phi)$ maps $S^2 \times S^2$ into S^1 ?

Theorem (C., Le Merdy, Potapov, Sukochev, Tomskova)

Let $M = \{m_{ikj}\}_{1 \leq i,k,j \leq N}$ be a family of complex numbers. Consider the bilinear Schur multiplication

$$T_M : S^2_N \times S^2_N \to S^1_N.$$

Theorem (C., Le Merdy, Potapov, Sukochev, Tomskova)

Let $M = \{m_{ikj}\}_{1 \leq i,k,j \leq N}$ be a family of complex numbers. Consider the bilinear Schur multiplication

$$T_M : S_N^2 \times S_N^2 \to S_N^1.$$

Then $\|T_M\| < 1$ if and only if there exist a Hilbert space K and two families $\{a_{ik}\}_{1 \leq i,k \leq N}$ and $\{b_{jk}\}_{1 \leq j,k \leq N}$ of elements of K such that

$$m_{ikj} = \langle a_{ik}, b_{jk} \rangle , \quad i,k,j \geq 1$$

and

$$\|a_{ik}\| < 1 \quad \text{and} \quad \|b_{jk}\| < 1, \quad i,k,j \geq 1.$$

Let A, B, C be normal operators on a separable Hilbert space H.

Main Theorem

For any $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$, *the following are equivalent :*
(*i*) $\Gamma^{A,B,C}(\phi)$ *is a bounded bilinear map* $S^2(H) \times S^2(H) \to S^1(H)$.

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(i) $\Gamma^{A,B,C}(\phi)$ is a bounded bilinear map $S^2(H) \times S^2(H) \to S^1(H)$.
(ii) There exist a Hilbert space K and two functions

$$a \in L^\infty(\lambda_A \times \lambda_B; K) \quad \text{and} \quad b \in L^\infty(\lambda_B \times \lambda_C; K)$$

such that

$$\phi(r, s, t) = \langle a(r, s), b(s, t) \rangle \quad \text{for a.e. } (r, s, t).$$

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In this case,

$$\|\Gamma^{A,B,C}(\phi) : S^2(H) \times S^2(H) \to S^1(H)\| = \inf \|a\|\|b\|.$$

Let $f:\mathbb{R}\to\mathbb{R}$ be a $\mathcal{C}^2$-function such that f'' is bounded.
Let $f^{[2]}:\mathbb{R}^3\to\mathbb{R}$ be defined by

$$f^{[2]}(x,y,z) := \begin{cases} \frac{f^{[1]}(x,y)-f^{[1]}(y,z)}{x-z}, & \text{if } x\neq z \\ \dfrac{\partial}{\partial x} f^{[1]}(x,y), & \text{if } x=z \end{cases}.$$

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Proposition

$$f(A + D) - f(A) - \frac{d}{dt}(f(A + tD))_{|t=0} = [\Gamma^{A+D,A,A}(f^{[2]})](D, D).$$

Thank you for your attention !