

# DIFFERENTIABILITY OF OPERATOR FUNCTIONS IN SCHATTEN NORMS

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- Let  $\mathcal{H}$  be a separable Hilbert space and let  $\mathcal{K}(\mathcal{H})$  be the set of compact operators on  $\mathcal{H}$ . We define, for  $1 \leq p < \infty$ ,

$$\mathcal{S}^p(\mathcal{H}) = \left\{ T \in \mathcal{K}(\mathcal{H}) \mid \|T\|_p = \text{tr}(|T|^p)^{1/p} < \infty \right\}.$$

Then  $(\mathcal{S}_p(\mathcal{H}), \|\cdot\|_p)$  is a Banach space called Schatten class of order  $p$ .

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Then  $(\mathcal{S}_p(\mathcal{H}), \|\cdot\|_p)$  is a Banach space called Schatten class of order  $p$ .

- Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a Lipschitz function. If  $1 < p < \infty$  and  $A, B$  are selfadjoint operators on  $\mathcal{H}$  such that  $A - B \in \mathcal{S}^p(\mathcal{H})$  then

$$f(A) - f(B) \in \mathcal{S}^p(\mathcal{H})$$

(D. Potapov, F. Sukochev, 2011)

Let  $1 < p < \infty$ , let  $A, K$  be selfadjoint operators on  $\mathcal{H}$  such that  $K$  is bounded. Define

$$\varphi : t \in \mathbb{R} \mapsto f(A + tK) - f(A) \quad (\in \mathcal{S}^p(\mathcal{H}) \text{ if } K \in \mathcal{S}^p).$$

Differentiability properties of  $\varphi$ ?

## Differentiability of first order

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- E. Kissin, D. Potapov, V. Shulman, F. Sukochev (2012)

If  $f$  is differentiable with bounded derivative, then  $\varphi$  is  $\mathcal{S}^p$ -differentiable.



## Higher order derivatives

- V. Peller (2006)

If  $f \in B_{\infty,1}^1(\mathbb{R}) \cap B_{\infty,1}^n(\mathbb{R})$ ,  $\varphi$  is  $n$ -times differentiable in the operator norm.

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If  $f \in C^n(\mathbb{R})$  with bounded derivatives  $f', \dots, f^{(n)}$ , then  $\varphi$  is  $n$ -times continuously  $\mathcal{S}^2$ -differentiable on  $\mathbb{R}$ .

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If  $1 < p < \infty$  and  $f \in C^n(\mathbb{R})$  with bounded derivatives, then  $\varphi$  is  $n$ -times  $\mathcal{S}^p$ -differentiable on  $\mathbb{R}$ .

If, in addition,  $f^{(n)} \in C_0(\mathbb{R})$ , then  $f$  is  $n$ -times continuously Fréchet  $\mathcal{S}^p$ -differentiable at every selfadjoint operator.

Let  $A_1, \dots, A_n$  be  $n$  normal operators on a separable Hilbert space  $\mathcal{H}$  and let, for any  $1 \leq i \leq n$ ,  $\lambda_{A_i}$  be a scalar valued spectral measure for  $A_i$ .

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$$\lambda_{A_i}(\Delta) = 0 \iff E^{A_i}(\Delta) = 0.$$

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Let  $\mathcal{B}_{n-1}(S^2 \times \dots \times S^2 \rightarrow S^2)$  be the space of bounded  $(n-1)$ -linear maps from the product of  $(n-1)$  copies of  $S^2(\mathcal{H})$  into  $S^2(\mathcal{H})$ .

Let

$$\Gamma^{A_1, A_2, \dots, A_n} : L^\infty(\lambda_{A_1}) \otimes \cdots \otimes L^\infty(\lambda_{A_n}) \longrightarrow \mathcal{B}_{n-1}(S^2 \times \cdots \times S^2 \rightarrow S^2)$$

be defined, for any  $f_i \in L^\infty(\lambda_{A_i})$  and for any  $X_1, \dots, X_{n-1} \in S^2$ , by

$$\begin{aligned} \left[ \Gamma^{A_1, A_2, \dots, A_n}(f_1 \otimes \cdots \otimes f_n) \right](X_1, \dots, X_{n-1}) = \\ f_1(A_1)X_1 f_2(A_2) \cdots f_{n-1}(A_{n-1})X_{n-1} f_n(A_n). \end{aligned}$$



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**THEOREM (C. C., C. LE MERDY, F. SUKOCHEV)**

$\Gamma^{A_1, A_2, \dots, A_n}$  extends to a  $w^*$ -continuous isometry

$$\Gamma^{A_1, A_2, \dots, A_n} : L^\infty \left( \prod_{i=1}^n \lambda_{A_i} \right) \longrightarrow \mathcal{B}_{n-1}(S^2 \times \cdots \times S^2 \rightarrow S^2).$$

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- When  $\mathcal{H}$  is finite dimensional, double operator integrals mappings behave like linear Schur multipliers and triple operator integrals behave like bilinear Schur multipliers.
- The theory of double operator integrals started with Birman-Solomyak, in a series of 3 papers published in 1966, 1967, 1973.

## Divided differences.

Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be differentiable. The divided difference of the first order  $f^{[1]} : \mathbb{R}^2 \rightarrow \mathbb{C}$  is defined by

$$f^{[1]}(x_0, x_1) := \begin{cases} \frac{f(x_0) - f(x_1)}{x_0 - x_1}, & \text{if } x_0 \neq x_1 \\ f'(x_0) & \text{if } x_0 = x_1 \end{cases}, \quad x_0, x_1 \in \mathbb{R}.$$

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If  $n \geq 2$  and  $f$  is  $n$ -times differentiable on  $\mathbb{R}$ , the divided difference of the  $n$ th order  $f^{[n]} : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$  is defined recursively by

$$f^{[n]}(x_0, x_1, \dots, x_n) := \begin{cases} \frac{f^{[n-1]}(x_0, x_2, \dots, x_n) - f^{[n-1]}(x_1, x_2, \dots, x_n)}{x_0 - x_1}, & \text{if } x_0 \neq x_1 \\ \partial_1 f^{[n-1]}(x_1, x_2, \dots, x_n) & \text{if } x_0 = x_1 \end{cases}.$$

If  $f^{(n)}$  is bounded then  $f^{[n]}$  is a bounded Borel function on  $\mathbb{R}^{n+1}$ .

## THEOREM

Let  $1 < p < \infty$ ,  $n \in \mathbb{N}$ ,  $n \geq 1$ ,  $f$  be  $n$ -times differentiable on  $\mathbb{R}$  with  $f^{(n)}$  bounded. Let  $A_1, \dots, A_{n+1}$  be selfadjoint operators in  $\mathcal{H}$ . Then

$$\Gamma^{A_1, A_2, \dots, A_{n+1}}(f^{[n]}) : S^{pn} \times \dots \times S^{pn} \rightarrow S^p$$

is bounded and there exists  $c_{p,n} > 0$  depending only on  $p$  and  $n$  such that,

$$\|\Gamma^{A_1, A_2, \dots, A_{n+1}}(f^{[n]})\| \leq c_{p,n} \|f^{(n)}\|_{\infty}.$$

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- D. Potapov, F. Sukochev, A. Skripka (2013) : case  $f \in C^n(\mathbb{R})$  and  $A_1 = A_2 = \dots = A_n$ .
- C. Le Merdy, A. Skripka (2018) : case  $f \in C^n(\mathbb{R})$ ,  $A_1, A_2, \dots, A_n$  distinct.
- C. C. (2019) : case  $f$   $n$ -times differentiable.



## THEOREM (C. C., 2019)

Let  $1 < p < \infty$ ,  $A$  and  $K$  be selfadjoint operators in  $\mathcal{H}$  with  $K \in \mathcal{S}^p(\mathcal{H})$ . Let  $n \in \mathbb{N}$ ,  $n \geq 1$  and let  $f$  be  $n$ -times differentiable on  $\mathbb{R}$  such that  $f^{(i)}$  is bounded for all  $1 \leq i \leq n$ . Consider the function

$$\varphi : t \in \mathbb{R} \mapsto f(A + tK) - f(A) \in \mathcal{S}^p(\mathcal{H}).$$

Then  $\varphi$  is  $n$ -times differentiable on  $\mathbb{R}$  and for any  $1 \leq k \leq n$ ,

$$\frac{1}{k!} \varphi^{(k)}(t) = \left[ \Gamma^{A+tK, A+tK, \dots, A+tK}(f^{[k]}) \right] (K, \dots, K), t \in \mathbb{R}.$$

In particular,  $\varphi^{(k)}$  is bounded on  $\mathbb{R}$ .

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In particular,  $\varphi^{(k)}$  is bounded on  $\mathbb{R}$ .

**Remark :** If  $A$  is bounded, we can assume that  $f$  is only  $n$ -times differentiable on  $\mathbb{R}$  with  $f^{(n)}$  locally bounded. In this case, the derivatives of  $\varphi$  are also locally bounded.

## PROPOSITION

Same assumptions on  $f$ . Let  $1 < p < \infty$ ,  $K$  be a selfadjoint operator in  $\mathcal{H}$  with  $K \in \mathcal{S}^{np}(\mathcal{H})$ . Denote

$$R_{n,p,A,K,f} = f(A + K) - f(A) - \sum_{k=1}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k} \left( f(A + tK) \right) \Big|_{t=0}.$$

Then,

$$R_{n,p,A,K,f} = \left[ \Gamma^{A+K,A,\dots,A}(f^{[n]}) \right] (K, \dots, K), \quad (1)$$

and we have the inequality

$$\|R_{n,p,A,K,f}\|_p \leq c_{p,n} \|f^{(n)}\|_\infty \|K\|_{np}^n. \quad (2)$$

## PROPOSITION

Let  $1 < p < \infty$ , let  $A$  and  $K$  be selfadjoint operators in  $\mathcal{H}$  with  $K \in \mathcal{S}^p(\mathcal{H})$ . Let  $n \geq 2$  and  $f \in C^n(\mathbb{R})$ . Assume that either  $A$  is bounded or  $f^{(i)}$  is bounded for all  $1 \leq i \leq n$ . Then the function

$$\varphi : t \in \mathbb{R} \mapsto f(A + tK) - f(A) \in \mathcal{S}^p(\mathcal{H})$$

is  $n$ -times continuously differentiable on  $\mathbb{R}$ .

## Sketch of the proof of the theorem for $A$ bounded

Case when  $p = 2$  and  $f \in C^n(\mathbb{R})$  :

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$$\frac{1}{t}(\varphi(t) - \varphi(0)) = [\Gamma^{A+tK, A}(f^{[1]})](K).$$

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For any  $u \in C_b(\mathbb{R})$ ,  $u(A + tK) \rightarrow u(A)$  strongly as  $t \rightarrow 0$ , from which we deduce that

$$[\Gamma^{A+tK, A}(f^{[1]})](K) \rightarrow [\Gamma^{A, A}(f^{[1]})](K) \text{ in } \mathcal{S}^2(\mathcal{H}).$$



## Case $1 < p < \infty$ and $f$ $n$ -times differentiable

**Step 1** : It is enough to prove the differentiability for  $K$  belonging to a dense subset  $F$  of  $\mathcal{S}^p(\mathcal{H})_{sa}$ .

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**Step 2 :** Take

$$F = \{i[A, Y] + Z \text{ with } Y, Z \in (\mathcal{S}^p(\mathcal{H}))_{sa} \text{ and } Z \text{ commutes with } A\}.$$

Let  $K = i[A, Y] + Z \in F$  and define  $\nu(t) = e^{-itY}(A + tZ)e^{itY}$ .  
We have  $\nu(0) = A$  and  $\nu'(0) = K$ .

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**Step 3 : simplification** For the case  $n = 1$  (proved by E. Kissin, D. Potapov, V. Shulman and F. Sukochev).

The functions  $t \mapsto f(A + tK) - f(A)$  and  $t \mapsto f(\nu(t)) - f(\nu(0))$  have the derivative in 0. But

$$f(\nu(t)) - f(\nu(0)) = e^{-itY} f(A + tZ)e^{itY} - f(A),$$

which we know how to differentiate.

For the higher order case, we use the following equality

$$\begin{aligned} & \left[ \Gamma^{A+K, A, \dots, A}(f^{[n-1]}) \right] (K, \dots, K) - \left[ \Gamma^{A, \dots, A}(f^{[n-1]}) \right] (K, \dots, K) \\ &= \left[ \Gamma^{A+K, A, \dots, A}(f^{[n]}) \right] (K, \dots, K). \end{aligned}$$

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combined with the previous steps...

Thank you !