Differentiability of operator functions and application to trace formula

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• Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{K}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$. We define, for $1 \leq p < \infty$,

$$\mathcal{S}^p(\mathcal{H}) = \left\{ T \in \mathcal{K}(\mathcal{H}) \mid \|T\|_p = \mathrm{tr}(|T|^p)^{1/p} < \infty \right\}.$$

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Then $(\mathcal{S}^p(\mathcal{H}), \|.\|_p)$ is a Banach space called Schatten class of order p.

• Let $f : \mathbb{R} \to \mathbb{C}$ be a Lipschitz function. If $1 < p < \infty$ and A, B are selfadjoint operators on $\mathcal{H}$ such that $A - B \in \mathcal{S}^p(\mathcal{H})$ then

$$f(A) - f(B) \in \mathcal{S}^p(\mathcal{H})$$

(D. Potapov, F. Sukochev, 2011)

Let $1 < p < \infty$, let A, K be selfadjoint operators on $\mathcal{H}$ such that K is bounded. Define

$$\varphi : t \in \mathbb{R} \mapsto f(A + tK) - f(A) \quad (\in \mathcal{S}^p(\mathcal{H}) \text{ if } K \in \mathcal{S}^p).$$

Differentiability properties of φ?

Differentiability of first order

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If $f \in C^2(\mathbb{R})$ and A is bounded, then φ is differentiable in the operator norm.

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• E. Kissin, D. Potapov, V. Shulman, F. Sukochev (2012)
If f is differentiable with bounded derivative, then φ is $\mathcal{S}^p$-differentiable.

Functions of Operators

Let $N \in \mathbb{N}, A = A^*, B = B^* \in M_N(\mathbb{C})$. Then,

$$A = \sum_{k=1}^{N} \lambda_k P_k \quad \text{and} \quad B = \sum_{j=1}^{N} \mu_j Q_j$$

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Let $f : \mathbb{R} \to \mathbb{C}$. We have

$$f(A) - f(B) \quad = \quad \sum_{k=1}^{N} f(\lambda_k) P_k - \sum_{j=1}^{N} f(\mu_j) Q_j$$

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$$\begin{aligned} f(A) - f(B) &= \sum_{k=1}^{N} f(\lambda_k) P_k - \sum_{j=1}^{N} f(\mu_j) Q_j \\ &= \sum_{k,j=1}^{N} \frac{f(\lambda_k) - f(\mu_j)}{\lambda_k - \mu_j} \underbrace{P_k (A - B) Q_j}_{X_{k,j}} \end{aligned}$$

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Schur multipliers

Let $M = (m_{ij}) \subset \mathbb{C}$ be a bounded family. Let

$$T_M : [a_{ij}] \mapsto [m_{ij} a_{ij}]$$

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Particular case : when $m_{ij} = f^{[1]}(i,j) = \dfrac{f(i) - f(j)}{i - j}$ where f is a Lipschitz function, then

$$\| T_M : \mathcal{S}^p \to \mathcal{S}^p \| \leq C \frac{p^2}{p-1} \|f\|_{Lip} \quad \text{(Potapov \& Sukochev, 2011)}$$

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Assume that $q \leq p$. The following are equivalent :

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1. $T_M : \mathcal{B}(\ell_p, \ell_q) \to \mathcal{B}(\ell_p, \ell_q)$ *is bounded.*
2. *There exist a measure space (a probability space when $p \neq q$) (Ω, μ) and two bounded sequences $(x_j)_j$ in $L^p(\mu)$ and $(y_i)_i$ in $L^{q'}(\mu)$ such that*

$$\forall i, j \in \mathbb{N}, m_{ij} = \langle x_j, y_i \rangle .$$

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$$\forall i, j \in \mathbb{N}, m_{ij} = \langle x_j, y_i \rangle .$$
3. *The mapping $u_M : \ell_1 \to \ell_\infty$ whose matrix is M has a factorization*

$$\begin{array}{ccc} \ell_1 & \xrightarrow{u_M} & \ell_\infty \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle S} \\ L^p(\mu) & \hookrightarrow & L^q(\mu) \end{array}$$

Multiple operator integrals

Let $A_1, \ldots, A_n$ be n normal operators on a separable Hilbert space $\mathcal{H}$ and let, for any $1 \leq i \leq n$, λ_{A_i} be a scalar valued spectral measure for A_i.

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It means that λ_{A_i} is a finite measure on $\sigma(A_i)$ such that, if E^{A_i} is the spectral measure of A_i then, for any borelian subset $\Delta \subset \sigma(A_i)$, we have

$$\lambda_{A_i}(\Delta) = 0 \Longleftrightarrow E^{A_i}(\Delta) = 0.$$

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Let $\mathcal{B}_{n-1}(S^2 \times \cdots \times S^2 \to S^2)$ be the space of bounded $(n-1)$-linear maps from the product of $(n-1)$ copies of $S^2(\mathcal{H})$ into $S^2(\mathcal{H})$.

Let

$$\Gamma^{A_1,A_2,\ldots,A_n} : L^\infty(\lambda_{A_1}) \otimes \cdots \otimes L^\infty(\lambda_{A_n}) \longrightarrow \mathcal{B}_{n-1}(S^2 \times \cdots \times S^2 \to S^2)$$

be defined, for any $f_i \in L^\infty(\lambda_{A_i})$ and for any $X_1, \ldots, X_{n-1} \in S^2$, by

$$\Big[\Gamma^{A_1,A_2,\ldots,A_n}(f_1 \otimes \cdots \otimes f_n)\Big](X_1, \ldots, X_{n-1}) = f_1(A_1)X_1 f_2(A_2) \cdots f_{n-1}(A_{n-1})X_{n-1} f_n(A_n).$$

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Theorem (C. C., C. Le Merdy, F. Sukochev, 2017)

$\Gamma^{A_1,A_2,\ldots,A_n}$ *extends to a* w^**-continuous isometry*

$$\Gamma^{A_1,A_2,\ldots,A_n} : L^\infty\left(\prod_{i=1}^n \lambda_{A_i}\right) \longrightarrow \mathcal{B}_{n-1}(S^2 \times \cdots \times S^2 \to S^2).$$

Operators of the form

$$\Gamma^{A_1,\dots,A_n}(\phi)$$

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If $A_i = \sum_{k=1}^{N} \lambda_k^i P_k^i \in M_N(\mathbb{C})$ and $\phi : \mathbb{R}^n \to \mathbb{C}$, we have

$$\left[\Gamma^{A_1,\ldots,A_n}(\phi)\right](X_1,\ldots,X_{n-1}) = \sum_{k_1,\ldots,k_n=1}^{N} \varphi(\lambda_{k_1}^1,\ldots,\lambda_{k_n}^n)\; P_{k_1}^1 X_1 P_{k_2}^2 \cdots P_{k_{n-1}}^{n-1} X_{n-1} P_{k_n}^n$$

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We have

$$f(A) - f(B) = \left[\Gamma^{A,B}(f^{[1]})\right](A-B).$$

Boundedness of Multiple operator integrals

Theorem (C.C., 2019)

Let $n \in \mathbb{N}, n \geq 2$, let $A_1, \ldots, A_n$ be normal operators on a separable Hilbert space $\mathcal{H}$ and let $\phi \in L^\infty(\lambda_{A_1} \times \cdots \times \lambda_{A_n})$. The following are equivalent :

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1. $\Gamma^{A_1,\ldots,A_n}(\phi)$ *extends to a (completely) bounded mapping*

$$\Gamma^{A_1,\ldots,A_n}(\phi)\colon \mathcal{K}(\mathcal{H}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H}).$$

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2. *There exist separable Hilbert spaces* $H_1, \ldots, H_{n-1}$,

$$a_1 \in L^\infty(\lambda_{A_1}; H_1), a_n \in L^\infty(\lambda_{A_n}; H_{n-1})$$

and

$$a_i \in L^\infty_\sigma(\lambda_{A_i}; \mathcal{B}(H_i, H_{i-1})), 2 \leq i \leq n-1,$$

such that

$$\phi(t_1, \ldots, t_n) = \langle a_1(t_1), [a_2(t_2) \ldots a_{n-1}(t_{n-1})](a_n(t_n)) \rangle \quad a.e.$$

Divided differences.

Let $f : \mathbb{R} \to \mathbb{C}$ be differentiable. The divided difference of the first order $f^{[1]} : \mathbb{R}^2 \to \mathbb{C}$ is defined by

$$f^{[1]}(x_0, x_1) := \begin{cases} \frac{f(x_0)-f(x_1)}{x_0-x_1}, & \text{if } x_0 \neq x_1 \\ f'(x_0) & \text{if } x_0 = x_1 \end{cases}, \qquad x_0, x_1 \in \mathbb{R}.$$

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If $n\geq 2$ and f is n-times differentiable on $\mathbb{R}$, the divided difference of the nth order $f^{[n]}:\mathbb{R}^{n+1}\to\mathbb{C}$ is defined recursively by

$$f^{[n]}(x_0,x_1,\dots,x_n):=\begin{cases}\frac{f^{[n-1]}(x_0,x_2,\dots,x_n)-f^{[n-1]}(x_1,x_2\dots,x_n)}{x_0-x_1}, & \text{if } x_0\neq x_1\\ \partial_1 f^{[n-1]}(x_1,x_2,\dots,x_n) & \text{if } x_0=x_1\end{cases}.$$

If $f^{(n)}$ is bounded then $f^{[n]}$ is a bounded Borel function on $\mathbb{R}^{n+1}$.

Theorem

Let $1 < p < \infty$, $n \in \mathbb{N}, n \geq 1$, f be n-times differentiable on $\mathbb{R}$ with $f^{(n)}$ bounded. Let $A_1, \ldots, A_{n+1}$ be selfadjoint operators in $\mathcal{H}$. Then

$$\Gamma^{A_1, A_2, \ldots, A_{n+1}}(f^{[n]}) : \mathcal{S}^{pn} \times \cdots \times \mathcal{S}^{pn} \to \mathcal{S}^p$$

is bounded and there exists $c_{p,n} > 0$ depending only on p and n such that,

$$\|\Gamma^{A_1, A_2, \ldots, A_{n+1}}(f^{[n]})\| \leq c_{p,n} \|f^{(n)}\|_\infty.$$

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- D. Potapov, F. Sukochev, A. Skripka (2013) : case $f \in C^n(\mathbb{R})$ and $A_1 = A_2 = \cdots = A_{n+1}$.
- C. Le Merdy, A. Skripka (2018) : case $f \in C^n(\mathbb{R}), A_1, A_2, \ldots, A_{n+1}$ distinct.
- C. C. (2019) : case f n-times differentiable.

Higher order derivatives

Theorem (C.C., C. Le Merdy, A. Skripka, F. Sukochev, 2018)

Let A and K be selfadjoint operators on $\mathcal{H}$ with $K \in \mathcal{S}^2(\mathcal{H})$. Let $n \in \mathbb{N}$ and $f \in C^n(\mathbb{R})$. Assume that either A is bounded or $f^{(i)}$ is bounded for all $1 \leq i \leq n$. Consider the function

$$\varphi : t \in \mathbb{R} \mapsto f(A + tK) - f(A) \in \mathcal{S}^2(\mathcal{H}).$$

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The function φ belongs to $C^n(\mathbb{R}, \mathcal{S}^2(\mathcal{H}))$ and for every integer $1 \leq k \leq n$ and $t \in \mathbb{R}$,

$$\frac{1}{k!}\varphi^{(k)}(t) = \left[\Gamma^{A+tK,A+tK,\ldots,A+tK}(f^{[k]})\right](K,\ldots,K).$$

Theorem (C. C., 2019)

Let $1 < p < \infty$, *A and K be selfadjoint operators in* $\mathcal{H}$ *with* $K \in \mathcal{S}^p(\mathcal{H})$. *Let* $n \in \mathbb{N}, n \geq 1$ *and let f be n-times differentiable on* $\mathbb{R}$ *such that* $f^{(i)}$ *is bounded for all* $1 \leq i \leq n$. *Consider the function*

$$\varphi : t \in \mathbb{R} \mapsto f(A + tK) - f(A) \in \mathcal{S}^p(\mathcal{H}).$$

Then φ *is n-times differentiable on* $\mathbb{R}$ *and for any* $1 \leq k \leq n$,

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In particular, $\varphi^{(k)}$ *is bounded on* $\mathbb{R}$.

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In particular, $\varphi^{(k)}$ is bounded on $\mathbb{R}$.

Remark : If A is bounded, we can assume that f is only n-times differentiable on $\mathbb{R}$ with $f^{(n)}$ locally bounded. In this case, the derivatives of φ are also locally bounded.

Proposition

Same assumptions on f. *Let* $1 < p < \infty$, K *be a selfadjoint operator in* $\mathcal{H}$ *with* $K \in \mathcal{S}^{np}(\mathcal{H})$. *Denote*

$$R_{n,p,A,K,f} = f(A+K) - f(A) - \sum_{k=1}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k}\Big(f(A+tK)\Big)\Big|_{t=0}.$$

Then,

$$R_{n,p,A,K,f} = \left[\Gamma^{A+K,A,\ldots,A}(f^{[n]})\right](K,\ldots,K), \tag{1}$$

and we have the inequality

$$\|R_{n,p,A,K,f}\|_p \leq c_{p,n}\|f^{(n)}\|_\infty \|K\|_{np}^n. \tag{2}$$

Proposition

Let $1 < p < \infty$, let A and K be selfadjoint operators in $\mathcal{H}$ with $K \in \mathcal{S}^p(\mathcal{H})$. Let $n \geq 2$ and $f \in C^n(\mathbb{R})$. Assume that either A is bounded or $f^{(i)}$ is bounded for all $1 \leq i \leq n$. Then the function

$$\varphi : t \in \mathbb{R} \mapsto f(A + tK) - f(A) \in \mathcal{S}^p(\mathcal{H})$$

is n-times continuously differentiable on $\mathbb{R}$.

Proposition

Let $1 < p < \infty$, let A and K be selfadjoint operators in $\mathcal{H}$ with $K \in \mathcal{S}^p(\mathcal{H})$. Let $n \geq 2$ and $f \in C^n(\mathbb{R})$. Assume that either A is bounded or $f^{(i)}$ is bounded for all $1 \leq i \leq n$. Then the function

$$\varphi : t \in \mathbb{R} \mapsto f(A + tK) - f(A) \in \mathcal{S}^p(\mathcal{H})$$

is n-times continuously differentiable on $\mathbb{R}$.

Theorem (C. Le Merdy, A. Skripka, 2018)

Let $1 < p < \infty$, let $n \in \mathbb{N}, n \geq 1$ and let $f \in C^n(\mathbb{R})$ be such that $f', \dots, f^{(n)}$ are bounded. Further assume either that $f^{(n)} \in C_0(\mathbb{R})$ or that $f^{(n+1)}$ exists and is bounded.
Then f is n-times continuously Fréchet-differentiable at every $A = A^*$.

Sketch of the proof of the theorem for A bounded

Case when $p = 2$ and $f \in C^n(\mathbb{R})$:

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Case $1 < p < \infty$ and f n-times differentiable

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Let $K = i[A, Y] + Z \in F$ and define $\nu(t) = e^{-itY}(A + tZ)e^{itY}$.
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Step 3 : simplification For the case $n = 1$ (proved by E. Kissin, D. Potapov, V. Shulman and F. Sukochev).
The functions $t \mapsto f(A + tK) - f(A)$ and $t \mapsto f(\nu(t)) - f(\nu(0))$ have the same derivative in 0. But

$$f(\nu(t)) - f(\nu(0)) = e^{-itY} f(A + tZ)e^{itY} - f(A),$$

which we know how to differentiate.

For the higher order case, we use the following equality

$$\left[\Gamma^{A+K,A,\dots A}(f^{[n-1]})\right](K,\dots,K) - \left[\Gamma^{A,\dots,A}(f^{[n-1]})\right](K,\dots,K)$$
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Then (V. Peller, 2016) there exists a function $\xi \in L^1(\mathbb{R})$ such that

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For example, if $\alpha \in \mathbb{C} \setminus \mathbb{R}$,

$$\mathrm{Tr}((A+K-\alpha I)^{-1} - (A-\alpha I)^{-1}) = -\int_{\mathbb{R}} \frac{1}{(t-\alpha)^2}\xi(t)\,\mathrm{d}t.$$

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Let $f \in C^2(\mathbb{R})$ be such that f'' is bounded. Assume that there exist a separable Hilbert space H and two bounded Borel functions $a, b : \mathbb{R}^2 \to H$ such that

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If, in addition, either f' is bounded or A is bounded, then

$$Tr\Big(f(A+K) - f(A) - \frac{d}{dt}f(A+tK)_{|t=0}\Big) = \int_{\mathbb{R}} f''(t)\eta(t)\,dt.$$

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Is this result optimal ?

A few questions...

1. If $A = A^* \in \mathcal{S}^2(H), U \in \mathcal{B}(H)$ is unitary, $f : \mathbb{T} \to \mathbb{C}$ with f' bounded, is

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$$\Gamma^{A_1,\ldots,A_n}(\phi) : \mathcal{S}^p \times \cdots \times \mathcal{S}^p \to \mathcal{S}^p$$

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3. Replace $\mathcal{S}^p$ by a non-commutative L^p-space.

Thank you !