Continuous linear and bilinear Schur multipliers and applications to perturbation theory

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1. Motivations
2. Linear Schur multipliers
3. Bilinear Schur multipliers
4. Multiple operator integrals and perturbation theory
5. S^1 and complete boundedness of triple operator integrals

Notations :

- For an operator A on a Hilbert space H, we denote by $\sigma(A) \subset \mathbb{C}$ its spectrum.
- $\mathcal{B}(X, Y)$ is the space of all continuous linear operators
$$T : X \to Y.$$
- An operator $T \in \mathcal{B}(\ell_p, \ell_q)$ will be identified with its infinite matrix $(t_{ij})_{i,j\geq 1}$.
- We denote by $S^p(H, K)$ the Schatten class of order p and by S_n^p their finite dimensional versions.

- Let $A = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ be a diagonal operator with $\lambda_i \in \mathbb{R}$ and let B be a selfadjoint operator on $\mathbb{C}^n$. For any $f \in C^1(\mathbb{R})$,

$$\frac{d}{dt}(f(A + tB))\Big|_{t=0} = D * B,$$

where $D * B$ is the Schur (or Hadamard) product of D and B, and D is the divided difference matrix

$$D_{ij} := \begin{cases} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}, & \text{if } \lambda_i \neq \lambda_j \\ f'(\lambda_i) & \text{if } \lambda_i = \lambda_j \end{cases}.$$

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so that

$$f(A+B) - f(A) - \frac{d}{dt}(f(A+tB))\Big|_{t=0} = B^2 \in S^1;$$

Definition

Let $M = (m_{ij})_{1 \leq i,j}$ be a family of complex numbers.
M is said to be a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$ (resp. on $\mathcal{S}^p$) if for any matrix $[a_{ij}] \in \mathcal{B}(\ell_p, \ell_q)$ (resp. in S^p),

$$M * A = [m_{ij} a_{ij}]$$

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- In 1977, Bennett gave a necessary and sufficient condition for a family M to be a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$.
- In 1985, V. Peller described more generally the Schur multipliers in the continuous case : the Schur multipliers on $\mathcal{B}(L_2)$.

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For $f \in L^{p'}(\Omega_1)$ and $g \in L^q(\Omega_2)$, we denote by $f \otimes g$ the mapping

$$\begin{aligned} f \otimes g : L^p(\Omega_1) &\longrightarrow L^q(\Omega_2) \\ h &\longmapsto \langle h, f \rangle\, g. \end{aligned} \tag{1}$$

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Let

$$T_\phi : L^{p'}(\Omega_1) \otimes L^q(\Omega_2) \to \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$$

be defined for any elementary tensor $f \otimes g \in L^{p'}(\Omega_1) \otimes L^q(\Omega_2)$ by

$$[T_\phi(f \otimes g)](h) = \left(\int_{\Omega_1} \phi(s, \cdot) f(s) h(s) \mathrm{d}\mu_1(s) \right) g(\cdot) \in L^q(\Omega_2),$$

for all $h \in L^p(\Omega_1)$.

Definition

We say that ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if there exists $C \geq 0$ such that for any $u \in L^{p'}(\Omega_1) \otimes L^q(\Omega_2)$,

$$\|T_\phi(u)\|_{\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))} \leq C\|u\|_\vee.$$

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Assume that $1 < p, q < +\infty$. If ϕ is a Schur multiplier, then the mapping

$$T_\phi : L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) \to \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$$

extends to w^*-continuous mapping, still denoted by

$$T_\phi : \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2)) \to \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$$

Let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. Denote by u_ϕ the mapping

$$\begin{array}{rccl} u_\phi : & L^1(\Omega_1) & \longrightarrow & L^\infty(\Omega_2). \\ & f & \longmapsto & \displaystyle\int_{\Omega_1} \phi(s,\cdot) f(s) \, \mathrm{d}\mu_1(s) \end{array}$$

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Theorem (C.)

Let $1 \leq q \leq p \leq +\infty$. Then ϕ is Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if and only if there exist a measure space (Ω, μ) (a probability space when $p \neq q$) and operators

$$R \in \mathcal{B}(L^1(\Omega_1), L^p(\Omega)), S \in \mathcal{B}(L^q(\Omega), L^\infty(\Omega_2))$$

such that

$$u_\phi = S \circ I \circ R,$$

where $I : L^p(\Omega) \to L^q(\Omega)$ is the inclusion mapping.

This property means that u_ϕ has the following factorization :

$$\begin{tikzcd} L^1(\Omega_1) \arrow[r, "u_\phi"] \arrow[d, "R"'] & L^\infty(\Omega_2) \\ L^p(\Omega) \arrow[r, hook, "I"'] & L^q(\Omega) \arrow[u, "S"'] \end{tikzcd}$$

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$\rightsquigarrow$ Notion of (p,q)-factorable operators $L_{p,q}(L^1(\Omega_1), L^\infty(\Omega_2))$.
This is a dual space :

$$L_{p,q}(L^1(\Omega_1), L^\infty(\Omega_2)) = (L^1(\Omega_1) \overset{\gamma_{p,q}}{\otimes} L^1(\Omega_2))^*.$$

Corollary

Let $M = (m_{ij})_{i,j\in\mathbb{N}} \subset \mathbb{C}$, $C \geq 0$ be a constant and let $1 \leq q \leq p \leq +\infty$. The following are equivalent :

(I) *M is a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$ with norm less than C.*

(II) *There exist a measure space (a probability space when $p \neq q$) (Ω, μ) and two bounded sequences $(x_j)_j$ in $L^p(\mu)$ and $(y_i)_i$ in $L^{q'}(\mu)$ such that*

$$\forall i,j \in \mathbb{N}, m_{ij} = \langle x_j, y_i \rangle \ \text{and} \ \sup_i \|y_i\|_{q'} \sup_j \|x_j\|_p \leq C.$$

An application : Inclusion relationships among spaces of Schur multipliers.
Let $\mathcal{M}(p,q)$ be the space of Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$.

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$$\mathcal{M}(q,r) \nsubseteq \mathcal{M}(p,p) \quad (\text{resp } \mathcal{M}(r,q) \nsubseteq \mathcal{M}(p,p)).$$

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(i) If $1 \leq r \leq q < p \leq 2$ *(resp.* $2 \leq p < q \leq r$*), then*

$$\mathcal{M}(q,r) \not\subseteq \mathcal{M}(p,p) \quad (resp\ \mathcal{M}(r,q) \not\subseteq \mathcal{M}(p,p)).$$

(ii) We have $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$ *if and only if* $1 \leq p \leq q \leq 2$ *or* $2 \leq q \leq p \leq +\infty$.

Definition

Let $M = \{m_{ikj}\}_{1\leq i,k,j\leq n}$ be a family of elements of $\mathbb{C}$. We define a bilinear Schur mapping by setting

$$T_M(A, B) := \sum_{1\leq i,j,k\leq n} m_{ikj} a_{ik} b_{kj} E_{ij},$$

for any $A = \{a_{ij}\}_{1\leq i,j\leq n}, B = \{b_{ij}\}_{1\leq i,j\leq n} \in M_n$.

Theorem (C., Le Merdy, Potapov, Sukochev, Tomskova)

Let $n \in \mathbb{N}$. Let $M = \{m_{ikj}\}_{i,k,j=1}^{n}$ be a family of complex numbers. For any $1 \leq k \leq n$, let $M(k)$ be the (classical) matrix given by $M(k) = \{m_{ikj}\}_{i,j=1}^{n}$. Then

$$\left\| T_M : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1 \right\| = \sup_{1 \leq k \leq n} \left\| T_{M(k)} : M_n \to M_n \right\|.$$

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Then $\|T_M\| < 1$ if and only if there exist a Hilbert space K and two families $\{a_{ik}\}_{1 \leq i,k \leq N}$ and $\{b_{kj}\}_{1 \leq j,k \leq N}$ of elements of K such that

$$m_{ikj} = \langle a_{ik}, b_{kj} \rangle \,, \quad i,k,j \geq 1$$

and

$$\|a_{ik}\| < 1 \;\text{ and }\; \|b_{kj}\| < 1, \quad i,k,j \geq 1.$$

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It means that λ_{A_i} is a finite measure on $\sigma(A_i)$ such that, if E^{A_i} is the spectral measure of A_i then, for any borelian subset $\Delta \subset \sigma(A_i)$, we have

$$\lambda_{A_i}(\Delta) = 0 \Longleftrightarrow E^{A_i}(\Delta) = 0.$$

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Let $\mathcal{B}_{n-1}(S^2 \times \cdots \times S^2 \to S^2)$ be the space of bounded $(n-1)$-linear maps from the product of $(n-1)$ copies of $S^2(\mathcal{H})$ into $S^2(\mathcal{H})$.

Let

$$\Gamma^{A_1,A_2,\ldots,A_n} : L^\infty(\lambda_{A_1})\otimes\cdots\otimes L^\infty(\lambda_{A_n}) \longrightarrow \mathcal{B}_{n-1}(S^2\times\cdots\times S^2 \to S^2)$$

be defined, for any $f_i \in L^\infty(\lambda_{A_i})$ and for any $X_1,\ldots,X_{n-1} \in S^2$, by

$$\Big[\Gamma^{A_1,A_2,\ldots,A_n}(f_1\otimes\cdots\otimes f_n)\Big](X_1,\ldots,X_{n-1}) = f_1(A_1)X_1 f_2(A_2)\cdots f_{n-1}(A_{n-1})X_{n-1}f_n(A_n).$$

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Theorem (C., Le Merdy, Sukochev)

$\Gamma^{A_1,A_2,\ldots,A_n}$ *extends to a* w^**-continuous isometry*

$$\Gamma^{A_1,A_2,\ldots,A_n} : L^\infty\left(\prod_{i=1}^n \lambda_{A_i}\right) \longrightarrow \mathcal{B}_{n-1}(S^2 \times \cdots \times S^2 \to S^2).$$

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$$\Gamma^{A_1,\dots,A_n}(\phi)$$

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• The theory of double operator integrals started with Birman-Solomyak, in a series of 3 papers published in 1966, 1967, 1973.

Let $f \in C^1(\mathbb{R})$. The divided difference of first order $f^{[1]} \colon \mathbb{R}^2 \to \mathbb{C}$ is defined by

$$f^{[1]}(x_0, x_1) := \begin{cases} \frac{f(x_0)-f(x_1)}{x_0-x_1} & \text{if } x_0 \neq x_1 \\ f'(x_0) & \text{if } x_0 = x_1 \end{cases}, \qquad x_0, x_1 \in \mathbb{R}.$$

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Let $n \geq 2$ and $f \in C^n(\mathbb{R})$. The divided difference of n-th order $f^{[n]} \colon \mathbb{R}^{n+1} \to \mathbb{C}$ is defined recursively by

$$f^{[n]}(x_0, x_1, \ldots, x_n) := \begin{cases} \frac{f^{[n-1]}(x_0, x_2, \ldots, x_n) - f^{[n-1]}(x_1, x_2 \ldots, x_n)}{x_0 - x_1} & \text{if } x_0 \neq x_1 \\ \partial_1 f^{[n-1]}(x_1, x_2, \ldots, x_n) & \text{if } x_0 = x_1 \end{cases},$$

for all $x_0, \ldots, x_n \in \mathbb{R}$.

Theorem (C., Le Merdy, Skripka)

Let A and K be selfadjoint operators on a separable Hilbert space $\mathcal{H}$ with $K \in \mathcal{S}^2(\mathcal{H})$. Let $n \geq 1$ and $f \in C^n(\mathbb{R})$. Assume either that A is bounded or that for all $1 \leq i \leq n$, $f^{(i)}$ is bounded. Let

$$\varphi : t \in \mathbb{R} \mapsto f(A + tK) - f(A) \in \mathcal{S}^2(\mathcal{H}).$$

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$$\varphi : t \in \mathbb{R} \mapsto f(A + tK) - f(A) \in \mathcal{S}^2(\mathcal{H}).$$

(*i*) *The function φ is n-times differentiable on $\mathbb{R}$ and for any integer $1 \leq \ell \leq n$ and any $t \in \mathbb{R}$,*

$$\frac{1}{\ell!}\varphi^{(\ell)}(t) = \left[\Gamma^{A+tK, A+tK, \ldots, A+tK}(f^{[\ell]})\right](K, \ldots, K).$$

Theorem

(*ii*) *We have*

$$f(A+K)-f(A)-\sum_{k=1}^{n-1}\frac{1}{k!}\varphi^{(k)}(0)$$
$$=\left[\Gamma^{A+K,A,\ldots,A}(f^{[n]})\right](K,\ldots,K).$$

Peller's problems : *Suppose that* $f \in C^2(\mathbb{R})$ *is such that* f'' *is bounded. Let* A *be a self-adjoint (possibly unbounded) operator and let* K *be a self-adjoint operator from* $\mathcal{S}^2$*. Is it true that*

$$\Gamma(A, K, f) := f(A+K) - f(A) - \frac{d}{dt}\Big(f(A+tK)\Big)\Big|_{t=0} \in \mathcal{S}^1?$$

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$\rightsquigarrow$ Replace $\Gamma(A, K, f)$ by $\left[\Gamma^{A+K,A,A}(f^{[2]})\right](K, K)$.

IF A and K are of the form $A = \oplus_{n\geq 1} A_n$, $K = \oplus_{n\geq 1} K_n$ where $A_n, K_n \in \mathcal{B}(\mathcal{H}_n)$ with $\mathcal{H}_n$ finite dimensional, then we have

$$\Gamma(A, K, f) = \bigoplus_{n\geq 1} \left(f(A_n + K_n) - f(A_n) - \frac{d}{dt}(f(A_n + tK_n))\Big|_{t=0} \right).$$

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Theorem (C., Le Merdy, Potapov, Sukochev, Tomskova)

There exist an unbounded self-adjoint operator A and a self-adjoint operator $K \in \mathcal{S}^2(\ell^2)$ such that

$$\Gamma(A,K,f) = f(A+K) - f(A) - \frac{d}{dt}(f(A+tK))\Big|_{t=0} \notin \mathcal{S}^1.$$

Lemma

There exist $C > 0$ and two sequences of operators $A_n, K_n \in B(\mathbb{C}^{8n+4})$ such that $\|K_n\|_2^2 \leq \frac{1}{n \ \log^{3/2} n}$, for all $n \geq N$, and

$$\|f(A_n + K_n) - f(A_n) - \frac{d}{dt}(f(A_n + tK_n))\big|_{t=0}\|_1 \geq \frac{C}{n\log^{1/2}(n)}.$$

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Define $f_0(x) = |x|$.

Theorem (Davies, 1988)

There exists a constant $C > 0$ such that for any $n \geq 1$, there exist $A_n, B_n \in \mathcal{B}(\mathbb{C}^{2n})$ self-adjoint such that $B_n \neq 0$ and

$$\|f_0(A_n + B_n) - f_0(A_n)\|_1 \geq C \log n \|B_n\|_1.$$

Can we characterize the functions ϕ for which $\Gamma^{A,B,C}(\phi)$ maps $S^2 \times S^2$ into the trace class S^1 ?

Theorem (C., Le Merdy, Sukochev)

Let $\mathcal{H}$ be a separable Hilbert space, let A, B and C be normal operators on $\mathcal{H}$ and let $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$. The following are equivalent :

(i) $\Gamma^{A,B,C}(\phi) \in B_2(S^2(\mathcal{H}) \times S^2(\mathcal{H}), S^1(\mathcal{H}))$.

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(I) $\Gamma^{A,B,C}(\phi) \in B_2(S^2(\mathcal{H}) \times S^2(\mathcal{H}), S^1(\mathcal{H}))$.

(II) *There exist a separable Hilbert space H and two functions*

$$a \in L^\infty(\lambda_A \times \lambda_B; H) \qquad \text{and} \qquad b \in L^\infty(\lambda_B \times \lambda_C; H)$$

such that

$$\phi(t_1, t_2, t_3) = \langle a(t_1, t_2), b(t_2, t_3) \rangle$$

for a.e. $(t_1, t_2, t_3) \in \sigma(A) \times \sigma(B) \times \sigma(C)$.

Theorem (C., Le Merdy, Sukochev)

Let $\mathcal{H}$ be a separable Hilbert space, let A, B and C be normal operators on $\mathcal{H}$ and let $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$. The following are equivalent :

(i) $\Gamma^{A,B,C}(\phi) \in B_2(S^2(\mathcal{H}) \times S^2(\mathcal{H}), S^1(\mathcal{H}))$.

(ii) *There exist a separable Hilbert space H and two functions*

$$a \in L^\infty(\lambda_A \times \lambda_B; H) \qquad \text{and} \qquad b \in L^\infty(\lambda_B \times \lambda_C; H)$$

such that

$$\phi(t_1, t_2, t_3) = \langle a(t_1, t_2), b(t_2, t_3) \rangle$$

for a.e. $(t_1, t_2, t_3) \in \sigma(A) \times \sigma(B) \times \sigma(C)$.

In this case,

$$\|\Gamma^{A,B,C}(\phi) \colon S^2(\mathcal{H}) \times S^2(\mathcal{H}) \longrightarrow S^1(\mathcal{H})\| = \inf \|a\|_\infty \|b\|_\infty,$$

where the infimum runs over all pairs (a, b) satisfying (ii).

- Let E, F be two Banach spaces, let $\Gamma_2(E, F)$ to be the space $T : E \to F$ which factor through Hilbert space.

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Theorem (C., Le Merdy, Sukochev)

Let (Ω, μ) be a separable measure space and let E, F be two separable Banach spaces. Let $\phi \in L^\infty_\sigma(\Omega; \Gamma_2(E, F^*))$. Then there exist a separable Hilbert space H and two functions

$$\alpha \in L^\infty_\sigma(\Omega; B(E,H)) \qquad \text{and} \qquad \beta \in L^\infty_\sigma(\Omega; B(F,H))$$

such that $\|\alpha\|_\infty \|\beta\|_\infty \leq \|\phi\|_\infty$ and for any $(x,y) \in E \times F$,

$$\langle [\phi(t)](x), y \rangle = \langle [\alpha(t)](x), [\beta(t)](y) \rangle, \qquad \text{for a.e. } t \in \Omega. \tag{2}$$

Connection with the theorem :

Regard $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$ as an element of $L^\infty_\sigma(\lambda_B; L^\infty(\lambda_A \times \lambda_C)$ and associate

$$\psi \in L^\infty_\sigma(\lambda_B; \mathcal{B}(L^1(\lambda_A); L^\infty(\lambda_C)))$$

by

$$[\psi(s)](f) = \int_{\sigma(A)} \phi(r, s, \cdot) f(r) \, d\lambda_A(r)$$

for any $f \in L^1(\lambda_A)$.

Connection with the theorem :

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$$[\psi(s)](f) = \int_{\sigma(A)} \phi(r, s, \cdot) f(r) \, d\lambda_A(r)$$

for any $f \in L^1(\lambda_A)$.

Theorem (Second formulation)

The following are equivalent :

(i) $\Gamma^{A,B,C}(\phi) \in B_2(S^2(\mathcal{H}) \times S^2(\mathcal{H}), S^1(\mathcal{H}))$.

(ii) *The function ψ associated to ϕ belongs to*

$$L^\infty_\sigma(\lambda_B; \Gamma_2(L^1(\lambda_A); L^\infty(\lambda_C))).$$

Theorem (C.)

Let $\mathcal{H}$ be a separable Hilbert space, A, B, C be normal operators on $\mathcal{H}$ and let $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$. The following are equivalent :

(i) $\Gamma^{A,B,C}(\phi)$ *extends to a completely bounded mapping*

$$\Gamma^{A,B,C}(\phi) : \mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}) \to \mathcal{S}^\infty(\mathcal{H}).$$

Theorem (C.)

Let $\mathcal{H}$ be a separable Hilbert space, A, B, C be normal operators on $\mathcal{H}$ and let $\phi \in L^\infty(\lambda_A \times \lambda_B \times \lambda_C)$. The following are equivalent :

(i) $\Gamma^{A,B,C}(\phi)$ *extends to a completely bounded mapping*

$$\Gamma^{A,B,C}(\phi) : \mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}) \to \mathcal{S}^\infty(\mathcal{H}).$$

(ii) *There exist a separable Hilbert space H, $a \in L^\infty(\lambda_A; H)$, $b \in L^\infty_\sigma(\lambda_B; \mathcal{B}(H))$ and $c \in L^\infty(\lambda_C; H)$ such that*

$$\phi(t_1, t_2, t_3) = \langle [b(t_2)](a(t_1)), c(t_3) \rangle$$

for a.e. $(t_1, t_2, t_3) \in \sigma(A) \times \sigma(B) \times \sigma(C)$.

In this case,

$$\left\| \Gamma^{A,B,C}(\phi) \right\| = \inf \|a\|_\infty \|b\|_\infty \|c\|_\infty. \qquad (3)$$

Thank you for your attention !